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CRACKING IN CONCRETE STRUCTURES -
CALCULATION OF CRACK WIDTH AND CRACK SPACING

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6. Keywords: cracking; restraint, serviceability; shrinkage; tensile creep; time-dependent

7. Abstract:

Cracks occur in reinforced concrete structures wherever and whenever the tensile stress in the concrete reaches the tensile strength of the concrete. After concrete sets and hardens, tensile stress at any location may be caused by factors such as early-age heat of hydration, applied loads, restrained shrinkage, temperature changes, settlement of the supports and so on. This report deals with the time-dependent development of stresses and strains before and after cracking in the tensile regions of reinforced concrete members and contains recommended procedures for the prediction of the spacing between cracks and the maximum crack width. Cracking caused by restraint to shrinkage in a variety of situations is considered, in addition to cracking caused by applied loads. The procedures are taken from Ref.1 which was prepared by the author and others for the Concrete Institute of Australia.

Cracking in concrete structures -
calculation of crack width and crack spacing

by

R. I. Gilbert

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1. INTRODUCTION

In many situations, cracking in reinforced concrete structures is inevitable. Cracks occur wherever and whenever the tensile stress in the concrete reaches the tensile strength of the concrete. After the concrete sets and hardens, tensile stress at any location may be caused by many different factors, including early-age heat of hydration, applied loads, restrained shrinkage, temperature changes, settlement of supports and so on. Cracks caused by restraint to load-independent deformation, including deformations due early-age cooling, shrinkage or ambient temperature changes, are termed intrinsic cracks. Cracks caused by applied loads are often called structural cracks. After cracking, further shrinkage induced deformation causes a significant increase in the width of cracks with time.

Many variables influence the width and spacing of intrinsic cracks, including the quantity, orientation and distribution of the reinforcement crossing the crack, the cover to the reinforcement, the bond characteristics of the reinforcement, the deformational properties of the concrete (including its creep and shrinkage characteristics), the environment and the size of the member. Considerable variations exist in the crack width from crack to crack and in the spacing between adjacent cracks, because of random variations in the properties of the in-situ concrete.

Control of cracking in concrete structures is often achieved by limiting the stress in the bonded reinforcement at the cracked section to some appropriately low value and ensuring that the bonded reinforcement is suitably distributed within the tensile zone. Building codes usually specify the maximum bar spacing for bonded reinforcement and the maximum concrete cover. Deterministic procedures for calculating crack widths are often specified, with the intention to control cracking by limiting the calculated crack width to some appropriately low value. However, the influence of shrinkage on crack widths is often not adequately considered and, as a consequence, excessively wide cracks in reinforced concrete structures are a relatively common problem.

The acceptance or rejection of concrete that has cracked, and the identification of appropriate repair methodologies, depends on a number of factors, but an adverse impact on aesthetics or appearance of concrete or its durability are possibly the main reasons. The assessment of cracking is very subjective and it is difficult to quantify the number and/or size of cracks that constitute unacceptable performance, as many unrelated factors play a part.

Cracks widths up to 0.3 mm wide are generally acceptable aesthetically, but specification of maximum crack width should consider the viewing distance and the type and significance of the structure. These factors have been combined in recommendations made in [Ref. 1] and are reproduced in Figure 1, where it is assumed that there are no structural or durability concerns with the cracking and that appearance is the sole concern. The acceptable crack width from a particular viewing distance reduces as the prestige of the structure increases and, for a given building, the acceptable crack width increases as the viewing distance increases. Figure 1 is provided here for a guide only. The selection of the maximum acceptable crack width on any concrete surface is a matter of engineering judgement.
Figure 1 can be used, together with an indication of the cost of designing for finer cracks, to help owners determine what design crack width is suitable for each structure when appearance is the only concern. Where cracks are hidden from view, including cracks at joints, aesthetics is not a consideration and wider crack widths may be acceptable if durability requirements are satisfied.

As there is no evidence to show that cracks widths below 0.5 mm increase the risk of corrosion damage in atmospheric exposures, it is recommended that a safe crack limit below this be adopted (see Table 1). There is relatively little guidance available on acceptable maximum crack widths. Eurocode 2 [Ref. 2] suggests limits of 0.3 mm to 0.4 mm under normal exposure conditions. Table 1 contains some recommended maximum final crack widths [Refs. 1] Serviceability and/or durability problems may arise if the final crack widths in the structure exceed these values.

It is important to appreciate that the values given in Table 1 are the total crack widths caused by early-age deformations, long-term deformations and loading. Where it is apparent that long-term effects will be additive to early age cracking then the limiting crack width associated with early deformations should be reduced accordingly. Where load induced tension and the contraction strains due to shrinkage and early-age cooling are in the same direction, there is a higher likelihood that the effects on crack spacing and crack width will be additive. When the load and shrinkage strains are perpendicular, they may be considered to act independently.

![Figure 1: Maximum specified crack width based on aesthetics requirements [Ref. 1]](image-url)
Table 1: Recommended maximum final design crack width, $w_{\text{max}}$ [Ref. 1]

<table>
<thead>
<tr>
<th>Environment</th>
<th>Design requirement</th>
<th>Maximum final crack width, $w_{\text{max}}$ (mm)</th>
</tr>
</thead>
</table>
| Sheltered environment: (where crack widths will not adversely affect durability) | Aesthetic requirement:  
- where cracking could adversely affect the appearance of the structure  
- where cracking will not be visible and aesthetics is not important | Owner and designer to determine based on guidance in Figure 1  
Designer to consider impact of crack width on shear strength |
| Exposed environment: (eg. exterior surfaces in above-ground locations) | Durability requirement:  
- where wide cracks could lead to corrosion of reinforcement | Reinforced concrete 0.3  
Prestressed concrete 0.2$^2$ |
| Aggressive environment: (eg. Surfaces of marine structures in or near seawater - spray zones, tidal/splash zones): | Durability requirement:  
- where wide cracks could lead to corrosion of reinforcement | Reinforced concrete  
0.30 (when $c^1 \geq 50$ mm)  
0.20 (otherwise),  
Prestressed concrete 0.2 |
| Water retaining and water excluding | Water tightness (with acceptable crack width depending on the requirements for leakage) | 0.2 or lower depending on hydrostatic head |

1 $c$ is the concrete cover to the nearest steel reinforcement.
2 For prestressed concrete in the interior of buildings where crack widths will not adversely affect durability throughout the design life, the limits for reinforced concrete apply.

This report contains a summary of recommended procedures for the control of cracking by the calculation of crack widths and crack spacings in a variety of design situations, including:

(i) shrinkage cracking in members with internal restraint provided by embedded reinforcement;
(ii) shrinkage cracking in elements with external end restraint;
(iii) shrinkage cracking in elements with external edge restraint
(iv) load induced cracking due to bending and axial tension.

The procedures are taken from Ref. 1 which was prepared by the author and others for the Concrete Institute of Australia.
2. PRINCIPLES OF CRACK FORMATION IN REINFORCED CONCRETE

2.1 Concrete Strain Components

Cracks form at a particular location in concrete when the concrete stress exceeds the tensile strength at that point. However, the prediction of the onset of cracking is not straightforward. The instantaneous tensile strength of concrete $f_{ct}(t_0)$ increases with the age of concrete at first loading $t_0$ and is a highly variable property of concrete. Under sustained tension, however, micro-cracking associated with tensile creep reduces the tensile strength and cracking may occur weeks or months after first loading at stress levels significantly less than $f_{ct}(t_0)$. The elastic modulus of the concrete also increases with time, $E_c(t)$. Restraint to shrinkage and cooling produces tensile stress in the concrete, even as the concrete shortens and, quite frequently, cracking occurs in concrete when the total concrete strain is compressive.

Under typical in-service conditions, the concrete strain at any point in a structure $\varepsilon_c(t)$ is the sum of a number of components, including the instantaneous (elastic) strain $\varepsilon_{ce}(t)$, the creep strain $\varepsilon_{cc}(t)$ and the shrinkage strain $\varepsilon_{cs}(t)$ (where $\varepsilon_{cs}(t)$ is the sum of the autogenous shrinkage $\varepsilon_{csa}(t)$ and drying shrinkage $\varepsilon_{csd}(t)$). If the temperature changes, there will also be a temperature component $\varepsilon_{cT}(t)$.

$$\varepsilon_c(t) = \varepsilon_{ce}(t) + \varepsilon_{cc}(t) + \varepsilon_{cs}(t) = \varepsilon_{ce}(t) + \varepsilon_{cc}(t) + \varepsilon_{free}(t)$$

(1)

The sum of the non-mechanical shrinkage and temperature strains is here referred to as the unrestrained or free shrinkage strain, $\varepsilon_{free}(t) = \varepsilon_{cT}(t) + \varepsilon_{cs}(t)$.

The elastic strain and the creep strain are related to stress and are produced by the external loads and by the restraining forces that develop as the concrete shrinks. The various strain components and their rates of change with time are illustrated in Figure 2a for a concrete specimen subjected to a constant sustained compressive stress $\sigma_{c0}$ first applied at time $t_0$. Figure 2b shows the corresponding strain components for an uncracked specimen subjected to a sustained tensile stress. In Figure 2, $t_d$ is the time when the shrinkage commences. If the sustained tension is high enough in Figure 2b, cracking may occur at some time after first loading when the total strain is compressive.

2.2 Restrained Shrinkage Cracking

Consider the unloaded, but fully restrained, reinforced concrete member shown in Figure 3a. The member is prevented from shortening at each end by rigid supports, so until the concrete cracks, the total concrete and steel strains are zero. Assuming the member remains uncracked, the development of the various strain components caused by early-age cooling and concrete shrinkage strains is shown in Figure 3b, where tensile strains are plotted below the time axis. The concrete free shrinkage strain $\varepsilon_{free}(t)$ commences at time $t_d$ and increases in magnitude with time at a decreasing rate. As the concrete shrinks, the restraining force $F(t)$ gradually increases producing tensile stress in the concrete (and tensile elastic and creep strains). The sum of the elastic and creep strains is the so-called restrained strain and, in a fully-restrained member, is equal and opposite to the shrinkage strain at any time before cracking.
Eventually, the first crack will occur on the weakest cross-section when the tensile stress $\sigma_c(t)$ ($= F_0(t)/A_c$) reaches the tensile strength of the concrete on that cross-section $f_{ct}$ (usually within a week of the commencement of drying in a fully-restrained members). The stress that develops in the concrete due to restraint to shrinkage is significantly reduced by tensile creep and tensile creep therefore delays the onset of first cracking. When the concrete cracks, the stress at the crack drops to zero and, therefore, so too does the tensile elastic strain. The net concrete strain becomes compressive and the magnitude of the restraining force drops as the crack opens to accommodate the shortening of the concrete.

Figure 3c shows the member immediately after the first crack has occurred and Figures 3d and 3e show the corresponding concrete and steel stresses in the restrained member. At first cracking, the restraining force reduces significantly from $A_c f_{ct}$ to $F_0$, depending on the amount of reinforcement, and the concrete stress away from the crack is less than the tensile strength of the concrete $f_{ct}$. The concrete on either side of the crack shortens elastically and the crack opens to a width $w(t)$ that depends on the area of reinforcement. At the crack, the steel carries the entire force $F_0$ and the stress in the concrete is zero. In the region adjacent to the crack (Region 2 in Figure 3d), the concrete and steel stresses vary considerably and the bond stress at the steel-concrete interface is high. At some distance $s_o$ on each side of the crack, the concrete and steel stresses are no longer influenced directly by the presence of the crack, as shown in Figures 3d and 3e (Region 1).
Figure 3: Concrete stress and strain history in an end-restrained member with concrete shrinkage [Ref 3].

The magnitude of $F_{i0}$ immediately after first cracking depends on the amount of steel in the member. If no steel is present, the restraining force drops to zero ($F_{i0} = 0$) and the crack opens widely, with the initial crack width equal to $w = \varepsilon_{cc}(t)L = f_{c}(t)L/E_{c}(t)$. The crack will subsequently become significantly wider as further shrinkage (and creep recovery) occurs in the now unrestrained member,
and no additional cracking can occur. However, if the area of steel is relatively large, cracking does not reduce the restraining force significantly and the crack width remains small. Further shrinkage will cause a gradual increase in $F_d(t)$ and additional cracks will develop with time. By including sufficient reinforcement, many cracks will develop in a restrained member, but each crack will be fine and well-controlled. Insufficient reinforcement may result in the steel yielding at the first crack and a single, wide and unserviceable crack will result.

As has been demonstrated, the amount and distribution of reinforcement greatly affect the spacing between cracks and the crack widths. Both crack spacing and crack width decrease as the area of reinforcement increases.

The member shown in Figure 3 is restrained at each end. Frequently, slabs and walls are restrained on one or more sides and this edge restraint may result in cracking perpendicular to the restrained edge. As is the case for end restraint, the width and spacing of edge restraint cracking depends on the quantity and distribution of reinforcement.

2.3 Load Induced Cracking – Due to Axial Tension

Consider the uniaxially loaded tension member shown in Figure 4a. Before cracking, the concrete tensile stress increases with increasing load. When the stress in the concrete first reaches the tensile strength at a particular section, a primary crack occurs at the applied load $P_{cr}$ (see Figure 4b). The first crack occurs at the weakest cross-section when the concrete tensile stress reaches the direct tensile strength of the section at the time of loading, $f_{ct}(t)$. The stress in the concrete at the crack drops to zero. The concrete stress increases with distance from the crack due to the steel-concrete bond, until at some distance $s_0$ from the crack, the concrete stress is no longer affected by the crack, as shown in Figure 4c. Slip at the concrete-steel interface in the region of significant bond stress (i.e. over the length $s_0$ on either side of the crack) causes the crack to open. The situation is similar to the direct tension cracking due to restrained shrinkage, except that the applied load $P$ may remain constant after cracking as it may not be dependent on member stiffness.

A relatively small increase in load will cause a second primary crack to develop at a cross-section at some distance $x \geq s_0$ from the first crack, thereby reducing the concrete stress in the vicinity of that crack. Eventually, under increasing load, primary cracks form at somewhat regular intervals along the member and the primary crack pattern is established. Primary cracks increase in width as the distance from the reinforcement bar increases, but tend to be more parallel sided at distances greater than 2 bar diameters from the bar. Under increasing load, it is reasonable to assume that the primary crack pattern will be established when the load $P$ reaches between $1.4P_{cr}$ and $1.8P_{cr}$. The concrete tensile stress at each primary crack is zero, rising to a maximum value $\sigma_{c1}$ (less than the tensile strength of the concrete) mid-way between adjacent cracks, as shown in Figure 4d. After the primary crack pattern is established, further increases in load may result in cover-controlled cracks developing between the primary cracks gradual breaking down the bond between the steel and the concrete and reducing tension stiffening still further (see Figure 4b). Under sustained load, the average tensile stress in the concrete gradually reduces primarily due to cracking and bond breakdown caused by drying shrinkage and, to a lesser extent, tensile creep.
Tension stiffening in a reinforced concrete member arises from the tensile stresses carried by the concrete between cracks. Tension stiffening contributes significantly to the stiffness of a member and is an important consideration when designing for deflection and crack control at the serviceability limit states. Tension stiffening is particularly significant in relatively lightly reinforced members, where the actual stiffness may be several times larger than the stiffness calculated on the basis of fully-cracked cross-sections, where tensile concrete is ignored and only the embedded tensile reinforcement is considered. Tension stiffening decreases when the average tensile stress in the concrete drops due to increases in either the magnitude or duration of the applied load.

### 2.4 Load Induced Cracking – Due to Bending

When the tensile strength of the concrete is reached at the extreme tensile fibre of a flexural member, primary flexural cracks develop at regular spacings on the tensile side of the member, as shown in Figure 5a. A sudden loss of stiffness occurs at first cracking and the short-term moment-curvature relationship becomes non-linear. The primary cracks penetrate spontaneously to a height $h_o$ (see Figure 5a) that depends on the quantity of steel and the magnitude of any axial force or prestress. For reinforced concrete members in pure bending with no axial force, the height of the primary cracks $h_o$ immediately after cracking is usually relatively high (0.6 to 0.9 times the depth of the member) and remains approximately constant under increasing bending moments until either the steel reinforcement yields or the concrete stress-strain relationship in the compressive region becomes non-linear. For prestressed members and members subjected to bending plus axial compression, $h_o$ may be relatively small initially and gradually increases as the applied moment increases, even when material behaviour is linear-elastic.
The stress in the tensile reinforcement and the stress in the concrete at the steel level for the cracked member are illustrated in Figures 5b and 5c, respectively. Immediately after first cracking, the intact concrete between adjacent primary cracks carries considerable tensile force, mainly in the direction of the reinforcement, due to the bond between the steel and the concrete. Over a gauge length containing several cracks, the average concrete tensile stress $\sigma_{c,av}$ at typical in-service levels of applied moment is a significant percentage of the tensile strength of the concrete. The steel stress is a maximum at a crack, where the steel carries the entire tensile force, and drops to a minimum between the cracks, as shown in Figure 5b. The bending stiffness of

![Diagram](image)

**Figure 5:** Stress distributions at the steel level in a cracked reinforced concrete member [Ref. 4]

the member is considerably greater than that based on a fully-cracked section, where concrete in tension is assumed to carry zero stress. This tension stiffening effect may be significant in the service-load performance of beams, and even more so for lightly reinforced slabs.

The Euler-Bernoulli assumption that plane sections remain plane is not strictly true for a cross-section in the cracked region of a beam. However, if strains are measured over a gauge length containing several primary cracks, the average strain diagram is linear over the depth of a cracked cross-section.

As the load increases above the cracking moment $M_{cr}$, and after the primary cracks have developed, secondary cracks (or cover-controlled cracks) form around the reinforcement between the primary cracks, the average concrete tensile stress drops and the tension stiffening effect gradually reduces. A typical moment versus average curvature relationship for a reinforced concrete cross-section in pure bending is shown in Figure 6a as the solid line OAB. A typical applied in-service moment in the post-cracking range is designated $M_s$ in the figure. Also shown in the figure as the dashed line OC of slope $(EI)_{cr}$ is the moment-curvature relationship for the fully-cracked cross-section. As moment increases after first cracking, the flexural rigidity gradually
reduces from that of the uncracked section \((EI)_{uncr}\) at first cracking and approaches that of the fully-cracked section \((EI)_{cr}\) as the moment becomes large, as shown.

If a flexural member, such as that shown in Figure 5a, begins to shrink prior to loading, as is commonly the case, a shrinkage induced curvature \(\kappa_{s,uncr}\) develops on the uncracked cross section when the applied moment is still zero (i.e. \(M_s = 0\)), shown in Figure 6b as point \(O'\). The restraint to shrinkage provided by the bonded reinforcement causes the tensile stress \(\sigma_s\) to develop with time in the concrete. The moment required to cause first cracking \(M_{cr,sh0}\) is less than \(M_{cr}\) as indicated in Figure 6b, where the moment curvature relationship is now represented by curve \(O'A'B'\). The initial curvature due to early shrinkage on a fully-cracked cross-section \((\kappa_{s,cr})\), where the cracked concrete is assumed to carry no tension, is significantly larger than that of the uncracked member \((\kappa_{s,uncr})\). Therefore, early shrinkage before loading causes the dashed line representing the fully-cracked response to move further to the right, shown as line \(O''C'\) in Figure 6b.

![Diagram](image)

**Figure 6:** Moment-average curvature relationship for a reinforced concrete cross-section at first loading, \(t_0\)

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For a prestressed concrete member, or a reinforced concrete member in combined bending and compression, the effect of tension stiffening is less pronounced because the loss of stiffness of the cracked section is less abrupt. As the applied moment increases, the depth of the primary cracks increases gradually (in contrast to the sudden crack propagation in a reinforced member in pure bending) and the depth of the concrete compressive zone is significantly greater than would be the case if no axial force was present.

As discussed in the previous section, tension stiffening is the contribution of the intact tensile concrete between the cracks to the post-cracking stiffness of the member. At each crack, the tensile concrete carries no stress, but as the distance from the crack increases, the tensile stress in the concrete increases due to the bond between the concrete and the tensile reinforcement. As the load increases, the average tensile stress in the concrete reduces as more cracks develop and, when the crack pattern is fully developed and the number of cracks has stabilized, the actual response becomes approximately parallel to the no tension response (OC in Figure 6a or O’′C’ in Figure 6b). For slabs containing small quantities of tensile reinforcement (typically in floor slabs $A_{st}/bd < 0.005$), tension stiffening may be responsible for more than 50% of the stiffness of the cracked member at service loads and $\delta \kappa$ (in Figure 6) remains significant up to and even beyond the point where the steel yields and the ultimate load is approached.

The keys to predicting the instantaneous deformation of a flexural member at service loads are firstly to evaluate the load required to cause first cracking or, more precisely, the moment to cause first cracking at the critical cross-section, and secondly to model tension stiffening accurately. Both of these tasks are not straightforward and must include consideration of the effects of shrinkage.

3. INCLUDING TENSILE CREEP AND SHRINKAGE IN CRACKED SECTION ANALYSIS

3.1 The Age-adjusted Effective Modulus Method (Refs. 4 to 7)

Consider the two concrete stress histories and the corresponding creep-time curves shown in Figure 7. In stress history (a), $\sigma_c(t_0)$ is applied at time $t_0$ and subsequently held constant with time. In stress history (b), the stress $\sigma_c(t)$ is gradually applied, beginning at $t_0$ and reaching a magnitude equal to $\sigma_c(t_0)$ at time $t_1$. The creep strain at any time $t (> t_0)$ produced by the gradually applied stress $\sigma_c(t)$ is significantly smaller than that resulting from the stress $\sigma_c(t_0)$ abruptly applied at $t_0$, as shown. This is due to aging. The earlier a concrete specimen is loaded, the greater is the final creep strain. The creep strain at time $t_1$ due to the constant stress history, $\varepsilon_{cc.a}(t_1)$, is the product of the elastic strain ($\sigma_c(t_0)/E_c$) and $\varphi_{cc}$, where $\varphi_{cc}$ is the creep coefficient at time $t_1$ due to a stress first applied at $t_0$. A reduced creep coefficient $\chi_{cc}\varphi_{cc}$ can be used to calculate the creep strain at $t_1$ caused by the gradually applied stress history (b), $\varepsilon_{cc.b}(t_1)$, where $\chi_{cc}$ is called the aging coefficient ($< 1.0$) first introduced by Trost (Ref. 5). The magnitude of the aging coefficient $\chi_{cc}$ generally falls within the range 0.4 to 1.0 depending on the rate of application of the gradually applied stress in the period $t_0$ to $t_1$. 

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Stress

(a) constant stress

\[ \sigma_c(t_0) \]

(b) gradually applied stress

\[ \sigma_c(t_0) \]

Time

Figure 7: Creep due to both constant and gradually applied stress histories.

Consider the gradually reducing stress history shown in Figure 8. An initial compressive stress \( \sigma_c(t_0) \) applied at time \( t_0 \), is reduced with time due to the application of a gradually increasing tensile stress increment \( \Delta \sigma_c(t) \). The change of stress may be due to a change of the external loads, or restraint to creep and shrinkage, or variations of temperature, or combinations of these, and is usually unknown at the beginning of an analysis.

The total strain at time \( t \) may be expressed using the AEMM as the sum of the strains produced by \( \sigma_c(t_0) \) (instantaneous and creep), the strains produced by the gradually applied stress increment \( \Delta \sigma_c(t) \) (instantaneous and creep), and the free shrinkage strain:

\[
\varepsilon(t) = [1 + \varphi_{cc}] \sigma_c(t_0)/E_c + [1 + \chi_{cc} \varphi_{cc}] \Delta \sigma_c(t)/E_c + \varepsilon_{free}(t)
\]

\[
= \sigma_c(t_0)/E_e + \Delta \sigma_c(t)/E_{ae} + \varepsilon_{free}(t)
\]

where \( E_e \) is the effective modulus and \( E_{ae} \) is the age-adjusted effective modulus given by:

\[
E_e = E_c/(1 + \varphi_{cc})
\]

\[
E_{ae} = E_c/(1 + \chi_{cc} \varphi_{cc})
\]

With the age-adjusted effective modulus method, two analyses need to be carried out: one at first loading (i.e. at time \( t_0 \)) and one at time \( t \) after the period of sustained stress. Eq. (2) is a stress-strain-time relationship for concrete that can be used to determine the time-dependent change in stress and deformation in a reinforced concrete member.

Figure 8: A gradually reducing stress history.
3.2  Restrained strain

Consider the strain at a point in an otherwise unloaded reinforced concrete structure in which a gradually increasing compressive strain \( \varepsilon_{\text{free}} \) develops in the concrete caused by early age cooling \( \varepsilon_{\text{cT}} \) and concrete shrinkage \( \varepsilon_{\text{cs}} \) \( \varepsilon_{\text{free}} = \varepsilon_{\text{cT}} + \varepsilon_{\text{cs}} \). If this concrete shortening is restrained either by the supports or by embedded reinforcement, a restraining force develops and the total concrete strain at any time can be expressed as:

\[
\varepsilon_c = \varepsilon_{\text{ce}} + \varepsilon_{\text{cc}} = (1 - R) \varepsilon_{\text{free}}
\]  

(5)

where the elastic and creep strains, \( \varepsilon_{\text{ce}} \) and \( \varepsilon_{\text{cc}} \), caused by the gradually increasing restraining force are both tensile and \( \varepsilon_{\text{free}} \) is of opposite sign. The sum of the elastic and creep strains is referred to as the restrained strain, \( \varepsilon_r = \varepsilon_{\text{ce}} + \varepsilon_{\text{cc}} = -R \varepsilon_{\text{free}} \), where \( R \) is the restraint factor which may vary between 0.0 and 1.0 depending on the nature and stiffness of the restraint. If the concrete is fully-restrained, such as in the end-restrained member of Figure 3, \( R = 1 \). If the member is restrained only by embedded reinforcement, \( R \) is less than unity depending on the area, location and orientation of the reinforcement. Determination of the restraint factor \( R \) is discussed subsequently.

If early-age cracking is being considered immediately after the heat of hydration cycle, \( \varepsilon_{\text{free}} \) is equal to the sum of the strain caused by cooling \( \varepsilon_{\text{cT}} \) and the autogenous shrinkage \( \varepsilon_{\text{cse}} \) at 3 days. If restraint cracking is being investigated at a later time \( t \), \( \varepsilon_{\text{free}} = \varepsilon_{\text{cT}} + \varepsilon_{\text{cs}} \), where \( \varepsilon_{\text{cs}} \) is the sum of the autogenous shrinkage \( \varepsilon_{\text{cse}} \) and drying shrinkage \( \varepsilon_{\text{csd}} \) strain components at that time.

Provided cracking has not occurred, the restrained strain can be expressed in terms of the gradually increasing tensile stress \( \Delta \sigma_r \) at the time under consideration. That is:

\[
\varepsilon_r = \varepsilon_{\text{ce}} + \varepsilon_{\text{cc}} = (\Delta \sigma_r / E_c) + \chi_{\text{cc}} \phi_{\text{cc}} (\Delta \sigma_r / E_c) = \Delta \sigma / E_{ae}
\]  

(6)

where \( \phi_{\text{cc}} \) is the tensile creep coefficient associated with the time period after the commencement of drying and \( \chi_{\text{cc}} \) is the corresponding aging coefficient introduced to account for the fact that \( \Delta \sigma \) is gradually applied to the concrete. Before cracking, therefore, the concrete stress caused by restraint \( \sigma_r \) is:

\[
\Delta \sigma = \varepsilon_r E_{ae} = -R \varepsilon_{\text{free}} E_{ae}
\]  

(7)

If cracking occurs, part of the average restrained strain (measured over a gauge length greater than the crack spacing) is relieved by the crack formation. This portion of the restrained strain is termed the crack-induced strain \( \varepsilon_{r,ct} \) and is important for the calculation of crack widths. The restrained stress at the crack is zero and the average tensile stress between the cracks is now:

\[
\Delta \sigma_{\text{av}} = (\varepsilon_r - \varepsilon_{r,ct}) E_{ae}
\]  

(8)

To avoid cracking resulting from early-age cooling and shrinkage, the tensile stress induced by restraint \( \Delta \sigma_r \) (Eq. 7) must be less than the mean tensile strength of the concrete at the time \( t \) under consideration, \( f_{\text{ctm}}(t) \). Even if restraint cracking does not occur, the stress \( \Delta \sigma_r \)
should not be ignored. Additional stresses caused by applied loads or by restraint to subsequent shrinkage and/or temperature strains may increase the tensile restraining stress to the tensile strength of the concrete and cause cracking. The design process for the control of cracking is the same, whether the tensile stress that results in cracking is caused by restraint or by applied loads and the control of such cracking requires an adequate quantity and distribution of steel reinforcement.

3.3 Restraint Factor due to early-age temperature differentials (internal restraint)

During hydration soon after setting, the temperature rises to its peak value at the centre of a member, dropping to its minimum value at the surfaces of the member. As cooling takes place, after the peak temperature has been reached, the interior of the member cools and contracts more than the surface and tension develops at the interior of the member, as shown in Figure 9. The concrete tensile stress caused by this strain differential is often sufficiently high to cause interior cracks to develop, as shown. The maximum tensile stress that develops at the interior of the member before cracking is

\[ \Delta \sigma = - \alpha_T \Delta T R E_{ae} \]  

where \( \alpha_T \) is the coefficient of thermal expansion for concrete, \( \Delta T \) is the maximum temperature drop, as shown in Figure 9. The restrained strain is calculated ignoring any simultaneous autogenous shrinkage which is uniform through the member thickness and does not contribute to the strain differential. In Eq. 9, \( \Delta T \) is negative, as it represents a drop in temperature. After cracking, the restrained stress drops, the crack-induced strain \( \varepsilon_{r,cr} \) increases and the cracks will open with time due to restrained drying and autogenous shrinkage. Shrinkage may cause the interior cracks that have developed during the heat of hydration cycle to extend to the surface and lead to full depth cracking.

![Figure 9: Development of strain, stress and possible cracking due to early-age cooling.](image)
At the interior of the member, where both the temperature drop and the restrained tensile stress are greatest, the restraint factor $R$ is readily determined by simple mechanics and depends on the temperature profile, which in turn depends on the mix characteristics, the member thickness and the environmental conditions. For a parabolic temperature profile, $R = 0.33$. For a triangular profile, $R = 0.5$. Bamforth [Ref. 8] recommended that $R$ is taken as 0.42.

### 3.4 Restraint Factor due to embedded reinforcement (internal restraint)

In this section, restraint factors are determined for various design situations in which the uncracked concrete and the bonded reinforcement undergo an initial cooling after the concrete sets ($\varepsilon_{cT}$ and $\varepsilon_{sT}$, respectively) and the concrete also undergoes autogenous (or chemical) shrinkage ($\varepsilon_{csa}$) and drying shrinkage ($\varepsilon_{csd}$). Provided the steel reinforcement has not yielded and the concrete is uncracked, the constitutive relationships for the concrete and the steel may be expressed as:

$$\varepsilon_c = \left( \frac{\sigma_c}{E_{ae}} \right) + \varepsilon_{cs} + \varepsilon_{cT} = \left( \frac{\sigma_c}{E_{ae}} \right) + \varepsilon_{\text{free}} \tag{10}$$

$$\varepsilon_s = \left( \frac{\sigma_s}{E_s} \right) + \varepsilon_{sT} \tag{11}$$

where $\varepsilon_{cs} = \varepsilon_{csa} + \varepsilon_{csd}$; $E_s$ is the elastic modulus of the steel; $E_{ae}$ is the age-adjusted effective modulus of concrete (Eq. 4); and $\varepsilon_{\text{free}} = \varepsilon_{cs} + \varepsilon_{cT}$.

#### 3.4.1 Symmetrically reinforced cross-sections:

Consider the unreinforced and unrestrained concrete member of length $L$ shown in Figure 10a and the symmetrically reinforced concrete member shown in Figure 10b. Except for the inclusion of longitudinal steel reinforcement of area $A_s$ symmetrically placed about the centroid of the cross-section in the second member, the two members are identical. A gradual compressive strain in the concrete caused by free shrinkage $\varepsilon_{\text{free}}$ would cause the unreinforced member to shorten by an amount $\varepsilon_{\text{free}} L$, as shown in Figure 10a.

\[
\Delta F_t = -(1-R)\varepsilon_{\text{free}} E_a A_s
\]

\[
\Delta F_t = R \varepsilon_{\text{free}} L
\]

(a) Unreinforced member  
(b) Symmetrically reinforced member

**Figure 10:** Restraint provided by symmetrically located reinforcement
The shortening deformation of the concrete in the reinforced member (Figure 10b), causes a gradual build-up of compression in the bonded reinforcement and this is opposed by an equal and opposite tensile force $\Delta F$ applied to the concrete. The gradually increasing tensile force results in tensile elastic strain and tensile creep strain, and the overall shortening of the member is reduced to $(1 - R) \varepsilon_{\text{free}} L$ (as shown in Figure 10b), where $R$ is the restraint factor ($0 \leq R \leq 1.0$), that depends on the amount of reinforcement. The compressive stress and force in the steel are

$$\sigma_s = (1 - R) \varepsilon_{\text{free}} E_s \quad (12)$$

$$F_s = (1 - R) \varepsilon_{\text{free}} E_s A_s \quad (13)$$

The reinforced concrete member is shortening, but the concrete is subjected to tensile force that could possibly result in, or contribute to, cracking.

Using the age-adjusted effective modulus method for the time-dependent analysis of the member in Figure 10b, and assuming compatibility of steel and concrete strains, it can be readily shown [Ref. 4] that the compressive concrete strain $\varepsilon_c$ (= $\varepsilon_c$) at time $t$ caused by the shrinkage strain $\varepsilon_{cs}$, the restraint factor $R$ and the concrete tensile stress $\sigma_c$ at time $t$ are given by:

$$\varepsilon_c = (1 - R) \varepsilon_{\text{free}} = \frac{(\varepsilon_{\text{free}} - \varepsilon_{st}) \alpha_e \rho}{(1 + \alpha_e \rho)} \varepsilon_{\text{free}} \quad (14)$$

$$R = 1 - \frac{(\varepsilon_{\text{free}} - \varepsilon_{st})}{\varepsilon_{\text{free}} (1 + \alpha_e \rho)} \quad (15)$$

$$\sigma_c = \frac{-(\varepsilon_{\text{free}} - \varepsilon_{st}) E_s \rho}{(1 + \alpha_e \rho)} \quad (16)$$

where $\alpha_e$ is the age adjusted modular ratio ( = $E_s/E_{ae}$); $\rho$ is the reinforcement ratio ( = $A_s/A_c$). The creep coefficient $\varphi_{cc}$ for use when determining the age-adjusted effective modulus of the concrete $E_{ae}$ (Eq. 4) is the tensile creep coefficient at time $t$ due to a stress first applied when contraction commenced. When considering the possibility of cracking after say 1 month, $\chi \varphi$ should not be taken greater than about 1.8.

When considering only restraint to shrinkage (i.e. when $\varepsilon_{ct} = \varepsilon_{st} = 0$), Eqs. 14 to 16 simplify to:

$$\varepsilon_c = (1 - R) \varepsilon_{cs} = \frac{\varepsilon_{cs}}{(1 + \alpha_e \rho)} \quad (17)$$

$$R = \frac{\alpha_e \rho}{(1 + \alpha_e \rho)} \quad (18)$$

$$\sigma_c = \frac{-\varepsilon_{cs} E_s \rho}{(1 + \alpha_e \rho)} \quad (19)$$
Consider a reinforced cross-section with $E_s = 200,000 \text{ MPa}$, $E_c = 20,000 \text{ MPa}$, $\chi_c \varphi_c = 1.625$, $\Delta T = -20^\circ \text{C}$ and $\varepsilon_s = -600 \times 10^{-6}$ (typical concrete properties for 30 MPa concrete when considering the possibility of restrained cracking at age 1 month). Taking the coefficients of thermal expansion for the concrete and steel as $\alpha_c = 10 \times 10^{-6}/\text{C}$ and $\alpha_s = 12 \times 10^{-6}/\text{C}$, respectively, the early age cooling strain in the concrete and the steel are $\varepsilon_c = \alpha_c \Delta T = -200 \times 10^{-6}$ and $\varepsilon_s = \alpha_s \Delta T = -240 \times 10^{-6}$, respectively. The free shrinkage strain in the concrete is therefore $\varepsilon_{\text{free}} = \varepsilon_c + \varepsilon_s = -800 \times 10^{-6}$.

The restraint factors $R$ determined using Eq. 15 are given in Table 2a for a wide range of reinforcement ratios. Also shown in Table 2a is the concrete tensile stress $\sigma_c$ induced by cooling and shrinkage (from Eq. 16).

The restraint factors $R$ determined using Eq. 18 are given in Table 2b for the case when $\Delta T = 0$ and $\varepsilon_{\text{free}} = \varepsilon_c = -600 \times 10^{-6}$. Also shown in Table 2a is the concrete tensile stress $\sigma_c$ induced by shrinkage only (from Eq. 19).

Even when these concrete stresses may not initiate cracking, such as when $\rho$ is less than about 3.0%, they will substantially reduce the applied load required to cause cracking.

Of course, this analysis assumes that the concrete is uncracked and that the tensile stress $\sigma_c$ can develop in the concrete. If any early-age cracking occurs at the end of the initial cooling period, the concrete in the vicinity of that crack cannot carry tension and the crack will open due to the subsequent shrinkage.

**Table 2**: Restraint factors and tensile stresses for symmetrically reinforced sections

(a) $\Delta T = -20^\circ \text{C}$ (with $\varepsilon_c = -600 \times 10^{-6}$; $\varepsilon_c = -200 \times 10^{-6}$; $\varepsilon_s = -240 \times 10^{-6}$)

<table>
<thead>
<tr>
<th>$\rho = A_s/A_c$</th>
<th>0.002</th>
<th>0.005</th>
<th>0.01</th>
<th>0.015</th>
<th>0.02</th>
<th>0.025</th>
<th>0.03</th>
<th>0.035</th>
<th>0.04</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>0.33</td>
<td>0.38</td>
<td>0.45</td>
<td>0.50</td>
<td>0.54</td>
<td>0.58</td>
<td>0.61</td>
<td>0.64</td>
<td>0.66</td>
</tr>
<tr>
<td>$\sigma_c$ (MPa)</td>
<td>0.21</td>
<td>0.50</td>
<td>0.89</td>
<td>1.21</td>
<td>1.47</td>
<td>1.69</td>
<td>1.88</td>
<td>2.04</td>
<td>2.19</td>
</tr>
</tbody>
</table>

(b) $\Delta T = 0$ (with $\varepsilon_c = -600 \times 10^{-6}$ and $\varepsilon_s = \varepsilon_s = 0$)

<table>
<thead>
<tr>
<th>$\rho = A_s/A_c$</th>
<th>0.002</th>
<th>0.005</th>
<th>0.01</th>
<th>0.015</th>
<th>0.02</th>
<th>0.025</th>
<th>0.03</th>
<th>0.035</th>
<th>0.04</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>0.05</td>
<td>0.12</td>
<td>0.21</td>
<td>0.28</td>
<td>0.34</td>
<td>0.40</td>
<td>0.44</td>
<td>0.48</td>
<td>0.51</td>
</tr>
<tr>
<td>$\sigma_c$ (MPa)</td>
<td>0.23</td>
<td>0.53</td>
<td>0.95</td>
<td>1.29</td>
<td>1.57</td>
<td>1.81</td>
<td>2.01</td>
<td>2.19</td>
<td>2.34</td>
</tr>
</tbody>
</table>

3.4.2 Unsymmetrically reinforced cross-sections:

If the reinforcement is not symmetrically placed on a cross-section, restraint to cooling and shrinkage contraction will induce a curvature on the section and a concrete tensile stress that may initiate cracking. Consider the singly-reinforced member shown in Figure 11a and the small segment of length, $\Delta z$. The shrinkage-induced stresses and strains on an uncracked and on a previously cracked cross-section are shown in Figures 11b and 11c, respectively.
As the concrete shrinks, the steel reinforcement is compressed and, in turn, the steel imposes an equal and opposite tensile force $\Delta F_t$ on the concrete at the level of the steel. This gradually increasing tensile force, acting at some eccentricity to the centroid of the concrete cross-section produces elastic and creep strains and a resulting curvature on the section. The shrinkage-induced curvature often leads to significant load independent deflection of the member. The magnitude of $\Delta F_t$ (and hence the shrinkage induced curvature) depends on the quantity and position of the reinforcement.

The curvature caused by $\Delta F_t$ obviously depends on the size of the (uncracked) concrete part of the cross-section, and hence on the extent of cracking, and this, in turn, depends on the magnitude of the applied moment and the quantity of reinforcement. For an uncracked singly-reinforced section, the restraint factor $R$, the restraining force $\Delta F_t$ and the extreme fibre concrete tensile stress $\sigma_c$ caused by a uniform non-mechanical concrete strain of magnitude $\varepsilon_{\text{free}} = \varepsilon_s + \varepsilon_{\text{cT}}$ may be approximated by [Ref. 4]:

$$R = - \frac{(\varepsilon_{\text{free}} - \varepsilon_{\text{cT}}) \alpha_{\text{se}} \rho (1 + \lambda_1)}{\varepsilon_{\text{free}} [1 + \alpha_{\text{se}} \rho (1 + \lambda_1)]}$$

(20)

Figure 11: Shrinkage-induced deformation and stresses in a singly-reinforced beam
\[
\Delta F_i = - \frac{(\epsilon_{\text{free}} - \epsilon_{\text{st}}) E_s A_s}{1 + \alpha_{cc} \rho (1 + \lambda_1)}
\]

(21)

\[
\sigma_c = - \frac{(\epsilon_{\text{free}} - \epsilon_{\text{st}}) E_s \rho (1 + \lambda_1 \lambda_2)}{1 + \alpha_{cc} \rho (1 + \lambda_1)}
\]

(22)

where the coefficients \(\lambda_1\) and \(\lambda_2\) depend on the geometry of the cross-section and are given by:

\[
\lambda_1 = 12 \left[ \left( \frac{d}{h} \right) - 0.5 \right]^2
\]

(23)

\[
\lambda_2 = 0.5h / (d - 0.5h)
\]

(24)

Consider an uncracked singly reinforced rectangular cross-section with \(d/h = 0.9\), \(E_s = 200,000\) MPa, \(E_c = 20,000\) MPa, \(\chi_{cc} \phi_{cc} = 1.625\), \(\Delta T = -20^\circ\), \(\epsilon_{cs} = -600 \times 10^{-6}\), \(\epsilon_{ct} = -200 \times 10^{-6}\) and \(\epsilon_{st} = -240 \times 10^{-6}\). The free shrinkage strain in the concrete is therefore \(\epsilon_{\text{free}} = \epsilon_{cs} + \epsilon_{ct} = -800 \times 10^{-6}\). The restraint factors \(R\) for the uncracked cross-section and the extreme fibre concrete tensile stresses \(\sigma_c\) have been determined using Eqs. 20 and 22 and are given in Table 3 for a wide range of reinforcement ratios \(\rho (=A_s/bh)\).

**Table 3**: Restraint factors \(R\) for an uncracked singly-reinforced cross-section \((d/h = 0.9)\)

<table>
<thead>
<tr>
<th>(\rho = A_s/bh)</th>
<th>0.002</th>
<th>0.004</th>
<th>0.006</th>
<th>0.008</th>
<th>0.01</th>
<th>0.012</th>
<th>0.014</th>
<th>0.016</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R)</td>
<td>0.09</td>
<td>0.16</td>
<td>0.22</td>
<td>0.27</td>
<td>0.30</td>
<td>0.34</td>
<td>0.36</td>
<td>0.39</td>
</tr>
<tr>
<td>(\sigma_c) (MPa)</td>
<td>0.66</td>
<td>1.17</td>
<td>1.57</td>
<td>1.89</td>
<td>2.16</td>
<td>2.38</td>
<td>2.57</td>
<td>2.74</td>
</tr>
</tbody>
</table>

For a cracked singly reinforced cross-section, the restraint factor is close to 1.0, as can be seen in Figure 11c. Table 4 shows the restraint factors \(R\) for the cracked singly-reinforced cross-section determined using the age-adjusted effective modulus method for a wide range of reinforcement ratios \(\rho (=A_s/bh)\), assuming in this case \(E_s = 200,000\) MPa, \(E_c = 20,000\) MPa, \(\chi_{cc} \phi_{cc} = 1.625\), \(\epsilon_{\text{free}} = \epsilon_{cs} = -600 \times 10^{-6}\), \(\epsilon_{ct} = -200 \times 10^{-6}\) and \(\epsilon_{st} = -240 \times 10^{-6}\). Also shown in Table 4 is the extreme fibre concrete tensile stress \(\sigma_c\) at the bottom of the uncracked part of the concrete in the compressive zone induced by the uniform shrinkage strain.

Although shrinkage strain is independent of stress, it appears that shrinkage curvature is not independent of external load. The shrinkage induced curvature on a previously cracked cross-section is considerably greater than on an uncracked cross-section, as can be seen in Figure 11.

**Table 4**: Restraint factor \(R\) for a cracked singly-reinforced cross-section \((d/h = 0.9)\)

<table>
<thead>
<tr>
<th>(\rho = A_s/bh)</th>
<th>0.002</th>
<th>0.004</th>
<th>0.006</th>
<th>0.008</th>
<th>0.01</th>
<th>0.012</th>
<th>0.014</th>
<th>0.016</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R)</td>
<td>0.987</td>
<td>0.966</td>
<td>0.942</td>
<td>0.916</td>
<td>0.891</td>
<td>0.865</td>
<td>0.841</td>
<td>0.817</td>
</tr>
<tr>
<td>(\sigma_c) (MPa)</td>
<td>0.485</td>
<td>0.676</td>
<td>0.808</td>
<td>0.908</td>
<td>0.986</td>
<td>1.047</td>
<td>1.096</td>
<td>1.135</td>
</tr>
</tbody>
</table>
3.5 Restraint Factor due to end restraint (external restraint)

In many situations, the movement of a reinforced concrete member is restrained at each end by the supports or by adjacent parts of the structure (such as the member shown in Figure 3). Such a member is said to be subject to end restraint. The member shown in Figure 12 is subjected to partial end-restraint. The supports suffer a relative displacement in the direction of the member axis $|\Delta u| < |\epsilon_{\text{free}} L|$, as shown, and the member shortens. The restraint factor is:

$$ R = 1 - \left[ \frac{\Delta u}{|\epsilon_{\text{free}} L|} \right] $$  \hspace{1cm} (25)

The magnitude of the restraint factor $R$ depends on the amount of movement that can occur at each end of the member. If the supports are immovable, the member is fully-restrained and the restraint factor $R$ is equal to 1.0. However, most supports will suffer some relative displacement $(\Delta u)$ and $R$ will be less than 1.0. In some situations, when adjacent parts of the structure are shrinking, the supports may move further apart $(\Delta u > 0)$ and the restraint factor will be greater than 1.0.

Due to concrete shrinkage $\epsilon_s$ (without considering early age cooling, i.e. $\epsilon_{\text{free}} = \epsilon_s$), an axial tensile restraining force $F_t$ develops in the member with time in the direction of the member axis:

$$ F_t = F_{ct} + F_{st} = -R \epsilon_s E_a A_c + (1 - R) \epsilon_s E_s A_s $$  \hspace{1cm} (26)

where $F_{ct}$ and $F_{st}$ are the forces carried by the concrete and the steel. If the member is fully-restrained (i.e. $R = 1.0$), the steel undergoes no strain and $F_{st} = 0$. If the member does shorten, $F_{ct}$ is tensile and $F_{st}$ is compressive.

If early-age cooling $(\epsilon_{CT}$ and $\epsilon_{ST})$ is considered together with shrinkage, the restraining force becomes:

$$ F_t = F_{ct} + F_{st} = -R \epsilon_{\text{free}} E_a A_c + [(1 - R) \epsilon_{\text{free}} - \epsilon_{CT}] E_s A_s $$  \hspace{1cm} (27)

where $\epsilon_{\text{free}} = \epsilon_s + \epsilon_{CT}$.

When the concrete stress caused by $F_t$ first reaches the tensile strength of the concrete $f_{ct}$ at a particular section, cracking occurs. The magnitude of $F_t$ after cracking and the crack width depend primarily on the amount of bonded reinforcement crossing the crack. If the member contains no longitudinal steel, cracking causes the restraining force $F_t$ to drop to zero and a wide,

![Figure 12: Full depth cracking in a member partially restrained at the ends.](image)
unsightly crack results. If the member contains only small quantities of reinforcement (for $\rho = A_s /bh$ less than about 0.002 for $f_{yk} = 500$ MPa), the steel at the crack yields, the crack opens widely and the restraining force drops to a relatively small fraction of its value prior to cracking. If the member contains relatively large quantities of reinforcement ($\rho > 0.008$ for $f_{yk} = 500$ MPa), the steel at each crack does not yield, the crack width remains small and, because the loss of member stiffness at cracking is not great, the restraining force remains high. Members containing large quantities of steel are therefore likely to suffer many cracks, but each will be fine and well controlled. For intermediate steel quantities ($0.002 < \rho < 0.008$ for $f_{yk} = 500$ MPa), cracking causes a loss of stiffness, an immediate reduction of $F_t$ and a crack width that may or may not be acceptable.

### 3.6 Restraint Factor due to edge restraint (external restraint)

Frequently, walls and slabs are subjected to edge restraint, where shortening due to early-age shrinkage is restrained on one or more sides of the element. Examples of edge restraint are shown again in Figure 13. In Figure 13a, contraction in the secondary direction of the one-way slab is restrained by the more massive supporting beams (contracting at a slower rate than the slab). The restraining forces applied to the slab along each supported edge cause a direct tension in the secondary direction of the slab that may cause the cracking shown in the isometric view. The spacing and width of these cracks depend on the amount and distribution of reinforcement in the secondary direction of the slab.

![Isometric view showing restrained shrinkage cracks.](image1)

(a) Contraction in the secondary direction of a one-way slab.

![Isometric view showing restrained shrinkage cracks](image2)

(b) Contraction in a wall restrained on one edge.

![Compression in beam at slab edge](image3)

![Plan view](image4)

![Elevation](image5)

![Buttress providing vertical restraint](image6)

![Footing providing horizontal restraint](image7)

(c) Contraction of wall with two adjacent edges restrained and potential cracking.

**Figure 13:** Typical edge restraint to contraction in walls and slabs [Ref. 1]
Figure 13b shows a wall where contraction $\varepsilon_{\text{free}}$ is restrained on one edge by the more massive footing. The restraint to contraction may result in cracking that is initiated at the base of the wall, where the resultant of the tensile restraint force is acting, and may extend the full height of the wall, as shown in the isometric view. A wall restrained on two adjacent edges is shown in Figure 13c, together with crack pattern initiated by restrained shrinkage in the two orthogonal directions.

The cracks resulting from restrained shrinkage are direct tension cracks caused by the tensile restraining force(s), $F_t$. They generally extend completely through the restrained slab or wall. If uncontrolled, these cracks can become unserviceable and lead to waterproofing and corrosion problems and may even compromise the integrity of the member.

Consider the wall and footing of Figure 13b and shown in cross-section in Figure 14a. Using the notation specified in the figure, the area, section modulus and second moment of area about the centroid of the wall are $A_1 = b_1 h_1$, $Z_1 = b_1 h_1^2/6$ and $I_1 = b_1 h_1^3/12$ and for the footing $A_2 = b_2 h_2$, $Z_2 = b_2 h_2^2/6$ and $I_2 = b_2 h_2^3/12$. Let us assume that the wall is cast at some time after the base, and so the wall will be cooling and shrinking faster than the base. Restrained cracking in the wall is to be checked at some time $t$ after casting. Let the free shrinkage strain in the wall be $\varepsilon_{\text{free.1}}$, while the contraction of the base during the same period is $\varepsilon_{\text{free.2}}$ (where $\varepsilon_{\text{free.2}} < \varepsilon_{\text{free.1}}$). The elastic modulus, creep coefficient, aging coefficient for the wall and base are $E_{c.1}$, $\varphi_{cc.1}$, $\chi_{cc.1}$ and $E_{c.2}$, $\varphi_{cc.2}$, $\chi_{cc.2}$, respectively. The corresponding age-adjusted effective moduli $E_{ae.1}$ and $E_{ae.2}$ are determined using Eq. 4. The self-equilibrating restraining forces that develop with time, $F_t$ (tensile) on the wall and $-F_t$ (compressive) on the base, act at a distance $y_t$ below the interface between the wall and the base, as shown in the elevation in Figure 14b. The longitudinal strains and stresses are shown in Figures 14c and 14d, respectively.

It can be readily shown that distance $y_t$ depends on the age-adjusted flexural rigidities of the wall and the base, $R_{aI.1} = E_{ae.1} I_1$ and $R_{aI.2} = E_{ae.2} I_2$, respectively, and is given by:

![Figure 14: Restraint actions and strains and stresses in an edge-restrained wall.](image-url)
The restraining force $F_t$ depends on the age-adjusted axial and first moment rigidities and the difference in free contraction of the wall and base, $\Delta \epsilon_{\text{free}} = \epsilon_{\text{free,1}} - \epsilon_{\text{free,2}}$ and may be obtained by enforcing strain compatibility at the wall-footing interface giving

$$F_t = \frac{-\Delta \epsilon_{\text{free}}}{\left(\frac{1}{R_{aA.1}} + \frac{1}{R_{aA.2}} + \frac{e_1}{R_{aB.1}} + \frac{e_2}{R_{aB.2}}\right)}$$  \hspace{1cm} (29)$$

where the terms $e_1$ and $e_2$ are the distances from the line of action of $F_t$ to the centroids of the wall and the base, respectively, i.e. $e_1 = (0.5h_1 + y_1)$; $e_2 = (0.5h_2 - y_1)$; $R_{aA.1} = E_{ae.1}A_1$; $R_{aA.2} = E_{ae.2}A_2$; $R_{aB.1} = E_{ae.1}Z_1$; and $R_{aB.2} = E_{ae.2}Z_2$.

The tensile stress at the bottom of the wall caused by the restraining force $F_t$ is:

$$\sigma_t = \frac{F_t}{A_1} + \frac{F_t(y_1 + 0.5h_1)}{Z_1}$$  \hspace{1cm} (30)$$

and the restrained strain $\varepsilon_t$ at this point and the corresponding restraint factor $R$ are:

$$\varepsilon_t = \sigma_t / E_{ae.1}$$  \hspace{1cm} (31)$$

$$R = \varepsilon_t / \Delta \epsilon_{\text{free}} = \sigma_t /(E_{ae.1}\Delta \epsilon_{\text{free}})$$  \hspace{1cm} (32)$$

**Example 1:**

Determine the stress and strain distribution due to the restraint to early-age contraction and shrinkage of the wall shown in Figure 15.

For the wall:

$$E_{c.1} = 20000 \text{ MPa}; \; \epsilon_{\text{free,1}} = -0.0006; \; \chi_{cc.1} \varphi_{cc.1} = 1.625;$$

and from Eq. 3: $E_{ae.1} = 7619 \text{ MPa}.$

For the footing:

$$E_{c.2} = 35000 \text{ MPa}; \; \epsilon_{\text{free,2}} = -0.0002; \; \chi_{cc.2} \varphi_{cc.2} = 0.975;$$

and from Eq. 3: $E_{ae.2} = 17,722 \text{ MPa}.$

With $b_1 = 200 \text{ mm}$ and $h_1 = 4000 \text{ mm}$ and $D_2 = 1000 \text{ mm}$ and $h_2 = 600 \text{ mm}$, the age-adjusted rigidities of the wall and the base are:
\( R_{aA.1} = E_{ae.1}A_1 = 6.095 \times 10^9 \text{ N}; \quad R_{aA.2} = E_{ae.2}A_2 = 1.06 \times 10^{10} \text{ N}; \)
\( R_{aB.1} = E_{ae.1}Z_1 = 4.063 \times 10^{12} \text{ Nmm}; \quad R_{aB.2} = E_{ae.2}Z_2 = 1.063 \times 10^{12} \text{ Nmm}; \)
\( R_{al.1} = E_{ae.1}I_1 = 8.127 \times 10^{15} \text{ Nmm}^2; \quad R_{al.2} = E_{ae.1}I_2 = 3.190 \times 10^{14} \text{ N mm}^2. \)

With \( \Delta \varepsilon_s = -0.0004, \quad e_1 = 2213.1 \text{ mm} \) and \( e_2 = 86.9, \) Eqs. 28 and 29 give:
\[
y_t = \frac{h_2 R_{al.1} - h_1 R_{al.2}}{2 (R_{al.1} + R_{al.2})} = 213.1 \text{ mm}
\]

and
\[
F_i = \frac{-\Delta \varepsilon_{\text{free}}}{\left( \frac{1}{R_{aA.1}} + \frac{1}{R_{aA.2}} + \frac{e_1}{R_{aB.1}} + \frac{e_2}{R_{aB.2}} \right)} = 452.3 \text{ kN}
\]

From Eqs. 30, 31 and 32, respectively:
\[
\sigma_c = F_i \frac{A_t}{h_t} = 2.44 \text{ MPa}
\]
\[
\varepsilon_t = \sigma_c / E_{ae.1} = +321 \times 10^{-6}
\]
\[
R = \frac{\varepsilon_t}{\Delta \varepsilon_{\text{free}}} = 0.801
\]

The longitudinal stress and strain distributions due to restrained contraction are shown in Figure 16.

**Figure 16:** Strain and stress distributions caused by contraction (Example 1)
4. CRACKS CAUSED BY RESTRAINT

4.1 End-restraint cracking in a beam, slab or wall due to shrinkage

Consider the fully-restrained member shown in Figure 17a. As the concrete shrinks, the restraining force $F_t$ gradually increases until the first crack occurs when $F_{t,\text{max}} = A_c f_{ct}$ (usually within a week of the commencement of drying and often even earlier during the cooling period soon after casting). At first cracking, the restraining force reduces to $F_{t,0}$, and the concrete stress away from the crack is less than the tensile strength of the concrete $f_{ct}$. The concrete on either side of the crack shortens elastically and the crack opens to a width $w_0$ that depends on the area of reinforcement (see Figure 17b). At the crack, the steel carries the entire force $F_{t,0}$ and the stress in the concrete is zero. In the region adjacent to the crack, the concrete and steel stresses vary considerably and the bond stress at the steel-concrete interface is high. At some distance $s_0$ on each side of the crack, the concrete and steel stresses are no longer influenced directly by the presence of the crack, as shown in Figures 17c and 17d.

---

**Figure 17:** First cracking in a restrained direct tension member [Ref. 3].
If the restraint is caused only by shrinkage of the concrete, numerical values of $F_{t0}$, $\sigma_{1,0}$, $\sigma_{s1,0}$, and $\sigma_{s2,0}$ can be obtained from Eqs. 33 to 36 [Ref.3]:

$$F_{t0} = \frac{\alpha_e \rho f_{ct} A_c}{C_1 + \alpha_e \rho (1 + C_1)}$$  \hspace{1cm} (33)

$$\sigma_{s1,0} = F_{t0} (1 + C_1) / A_c$$  \hspace{1cm} (34)

$$\sigma_{s2,0} = -C_1 \sigma_{s2,0}$$  \hspace{1cm} (35)

$$\sigma_{s2,0} = F_{t0} / A_s$$  \hspace{1cm} (36)

where $\alpha_e$ is the modular ratio ($E_s/E_c$), $\rho$ is the reinforcement ratio ($A_s/A_c$), $f_{ct}$ is the direct tensile strength of the concrete at first cracking and

$$C_1 = \frac{2s_o m}{3L - 2s_o m}$$  \hspace{1cm} (37)

In Eq. 37, $m$ is the number of cracks that have formed in the length $L$ and, of course, at first cracking $m = 1$.

If the restraint is caused by early age cooling of the steel and concrete, in addition to shrinkage of the concrete, numerical values of $F_{t0}$, $\sigma_{s1,0}$, $\sigma_{s1,0}$, and $\sigma_{s2,0}$ can be obtained from Eqs. 38 to 41:

$$F_{t0} = \frac{\alpha_e \rho f_{ct} A_c}{C_1 + \alpha_e \rho (1 + C_1)} - \frac{E_s A_s C_2 \alpha_e \rho (1 + \alpha_e \rho)}{C_1 + \alpha_e \rho (1 + C_1)}$$  \hspace{1cm} (38)

$$\sigma_{s1,0} = \frac{F_{t0} (1 + C_1)}{A_c} + C_2 E_s \rho$$  \hspace{1cm} (39)

$$\sigma_{s1,0} = -C_1 \sigma_{s2,0} - C_2 E_s$$  \hspace{1cm} (40)

$$\sigma_{s2,0} = F_{t0} / A_s$$  \hspace{1cm} (41)

where

$$C_2 = \frac{3\varepsilon_T L}{3L - 2s_o m}$$  \hspace{1cm} (42)

The distance $s_o$ over which stresses vary on either side of a crack depends on the bond stress at the steel-concrete interface, and this is influenced by the reinforcement quantity, the bar diameter $\phi$ and the surface characteristics of the bar. In Ref. 4, $s_o$ was taken to be:

$$s_o = 0.1 \phi \rho$$  \hspace{1cm} (43)

This expression was used earlier by Favre et al. [Ref. 9] and others for a member containing deformed bars or welded wire mesh. More recent experimental results [Ref. 10] suggest that shrinkage causes a deterioration in bond at the steel-concrete interface and a gradual increase in $s_o$ with time. For calculations at first cracking using (Eqs. 33 to 43), $s_o$ may be taken from Eq. 43,
and for final or long-term calculations, the final value of \( s_0 (s^*o) \) should be multiplied by 1.33 [Ref. 11]. That is:

\[
s^*o = 1.33 \, s_0 \tag{44}
\]

The final number of cracks and the final average crack width depend on the length of the member, the quantity and distribution of reinforcement, the quality of bond between the concrete and steel, the amount of shrinkage and the concrete properties. In Figure 18a, a portion of a restrained direct tension member is shown after all shrinkage has taken place and the final crack pattern is established. The average concrete and steel stresses caused by shrinkage are illustrated in Figures 18b and 18c.

By enforcing the requirements of compatibility and equilibrium, expressions for the average crack spacing \( s \) and crack width \( w \) in a restrained member due to concrete shrinkage have been derived by Nejadi and Gilbert [Ref. 11].

Providing the reinforcing steel has not yielded, equating the overall shortening of the steel reinforcement to \( \Delta u \) for a member containing \( m \) cracks gives:

\[
\frac{\sigma_{s1}}{E_s} L + m \frac{\sigma_{s2} - \sigma_{s1}}{E_s} \left( \frac{2}{3} s^*o + w \right) + \varepsilon_{st} L = \Delta u \tag{45}
\]

and, as \( w \) is much less than \( s_0 \), rearranging gives:

**Figure 18:** Final concrete and steel stresses after direct tension cracking [Ref. 3].
\[ \sigma_1 = -\sigma_2 C_1 - E_s C_2 + E_s C_3 \]  
where \( C_1 \) and \( C_2 \) are given by Eqs. 37 and 42, respectively, and

\[ C_3 = \frac{3\Delta u}{3L - 2s_o m} \]

(47)

At each crack:

\[ \sigma_{s2} = F_t / A_s \]  

(48)

In Region 1 in Figure 18, where the distance away from each crack exceeds \( s_o^* \) (with \( s_o^* \) estimated using Eq. 44 (and Eq. 43), the concrete stress history is shown diagrammatically in Figure 19. The concrete tensile stress increases gradually with time as shrinkage progresses and approaches the direct tensile strength of the concrete \( f_{ct} \). When cracking first occurs, the tensile stress in the uncracked regions drops suddenly as shown. Although the concrete stress history is continually changing, the average concrete stress at any time after first cracking, \( \sigma_{av} \), is between \( \sigma_{c1.0} \) and \( f_{ct} \), as shown in Figure 19, and may be taken as the average of \( \sigma_{c1.0} \) and \( f_{ct} \) [Ref. 3].

The final creep strain in Region 1 may be approximated by:

\[ \varepsilon_{cc1} = \varphi_{cc} (\sigma_{c.av} / E_c) \]  

(49)

where \( \varphi_{cc} \) is the tensile creep coefficient (at the time under consideration) for concrete first loaded at the age when the contraction first commenced.

The concrete strain in Region 1 is the sum of the elastic, creep, shrinkage and temperature components and may be approximated as:

\[ \varepsilon_1 = \varepsilon_{e1} + \varepsilon_{cc1} + \varepsilon_{s} + \varepsilon_{T} = \varepsilon_{e1} + \varepsilon_{cc1} + \varepsilon_{free} \]

\[ = (\sigma_{c} / E_c) + \varphi_{cc} (\sigma_{c.av} / E_c) + \varepsilon_{free} \]  

(50)

The magnitude of the tensile creep coefficient \( \varphi_{cc} \) after all the contraction due shrinkage and cooling is completed for concrete first loaded at the commencement of drying/cooling is usually between 3 and 4, depending on the quality of the concrete.

In Region 1, at any distance from a crack greater than \( s_o^* \), equilibrium requires that the sum of the force in the concrete and the force in the steel is equal to the restraining force \( F_t \). That is:

\[ \sigma_{c1} A_c + \sigma_{s1} A_s = F_t \quad \text{or} \quad \sigma_{c1} = (F_t - \sigma_{s1} A_s) / A_c = (F_t / A_c) - \sigma_{s1} \rho \]  

(51)

**Figure 19:** Concrete stress history in uncracked Region 1 [Ref. 4].
The compatibility requirement in Region 1 is that the final concrete and steel strains are identical (i.e. \( \varepsilon_{s1} = \varepsilon_{c1} \)). That is:

\[
(\sigma_s/E_s) + \varepsilon_{sT} = (\sigma_c/E_c) + \varepsilon_{cT} + \varepsilon_{\text{free}} \tag{52}
\]

Substituting Eqs. 46 and 48 into Eq. 52 and rearranging gives:

\[
F_t = \frac{(C_3 - C_2) E_s A_s (1 + \alpha_e \rho)}{1 + \alpha_e \rho (1 + C_1)} - \frac{(\varepsilon_{cT} + \varepsilon_{\text{free}} - \varepsilon_{sT}) E_s A_s}{1 + \alpha_e \rho (1 + C_1)} \tag{53}
\]

where \( \alpha_e \) is the effective modular ratio \( (E_s/E_c) \) and \( \varepsilon_{cT} \) is the final creep strain in region (Eq. 49).

With the restraining force \( F_t \) and the steel stress in Region 1 obtained from Eqs. 53 and 46, respectively, the final concrete stress in Region 1 is calculated from Eq. 51.

The number of cracks \( m \) is the lowest integer value of \( m \) for which \( \sigma_{c1} \leq f_{ct} \). The direct tensile strength \( f_{ct} \) should be taken as the mean 28 day value. The overall shortening of the concrete is an estimate of the sum of the crack widths. The final concrete strain at any point in Region 1 of Figure 18 is given by Eq. 50, and in Region 2, the final concrete strain is

\[
\varepsilon_{c2} = (f_n \sigma_{c1}/E_c) + \varepsilon_{\text{free}} \tag{54}
\]

where \( E_c \) is the effective modulus of concrete given by Eq. 3 and \( f_n \) varies between zero at a crack and unity at \( s_0^* \) from a crack. If a parabolic variation of stress is assumed in Region 2, the following expression for the average crack width \( w \) is obtained by integrating the concrete strain over the length of the member:

\[
w = -[(s - 0.667 s_0^*)(\sigma_{c1}/E_c) + \varepsilon_{\text{free}} s] \tag{55}
\]

The preceding analysis is valid provided the assumption of linear-elastic behaviour in the steel is valid, i.e. provided the steel has not yielded.

**Example 2:**

Consider the 140 mm thick slab of length 4 m shown in Figure 20. The slab is rigidly held in position at each end support and, except for restraint to shrinkage, is unloaded. The slab is symmetrically reinforced with 12mm diameter longitudinal bars at 250 mm centres near both the top and bottom surfaces. Hence, \( A_s = 900 \text{ mm}^2/\text{m} \) and \( \rho = A_s/A_c = 0.00643 \). The average spacing between the restrained shrinkage cracks and the average final crack width are to be determined.

The material properties are:

\[
f_{ct} = 2.5 \text{ MPa}; E_c = 25,000 \text{ MPa}; \quad \alpha_e = E_s/E_c = 8.0; \quad \varphi_{cK} = 2.5; \quad E_s = E_c/(1+\varphi_{cK}) = 7,143 \text{ MPa}; \quad \alpha_{ee} = E_s/E_e = 28; \quad \varepsilon_{s} = -0.0006; \quad \varepsilon_{cT} = -0.0002; \quad \varepsilon_{sT} = -0.00024; \quad \Delta u = 0; \quad E_s = 200,000 \text{ MPa and } f_{yk} = 500 \text{ MPa.}
\]
From Eqs. 43 and 44: \( s_0 = 0.1 \times 12 / 0.00643 = 187 \text{ mm} \) and \( s_0^* = 1.33 \times 187 = 248 \text{ mm} \)

From Eqs. 37 and 42:

\[
C_1 = \frac{2 \times 187}{3 \times 4000 - 2 \times 187} = 0.0321 \quad \text{and} \quad C_2 = \frac{3 \times (-0.00024) \times 4000}{3 \times 4000 - 2 \times 187} = -0.000248
\]

From Eqs. 38 and 39:

\[
F_{c1.0} = \frac{8 \times 0.00643 \times 2.5 \times 1000 \times 140}{0.0321 + 8 \times 0.00643 \times (1 + 0.0321)} = \frac{2 \times 10^5 \times 900 \times 2.5 \times 0.248^3 \times (1 + 8 \times 0.00643)}{0.0321 + 8 \times 0.00643 \times (1 + 0.0321)}
\]

\[= 239.6 \text{ kN} \]

\[
\sigma_{c1.0} = \frac{239.6 \times 10^3 (1 + 0.0321)}{1000 \times 140} = 1.45 \text{ MPa}
\]

and therefore: \( \sigma_{c \text{av}} = (\sigma_{c1.0} + f_{c1}) / 2 = 1.97 \text{ MPa} \)

The tabulation below shows the final restraining force and concrete and steel stresses for different numbers of cracks:

<table>
<thead>
<tr>
<th>Number of cracks, ( m )</th>
<th>( F ) from Eq. 53 (kN)</th>
<th>( \sigma_{c1} ) from Eq. 46 (MPa)</th>
<th>( \sigma_{c2} ) from Eq. 48 (MPa)</th>
<th>( \sigma_{c3} ) from Eq. 51 (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>376</td>
<td>-48.5</td>
<td>418</td>
<td>3.00</td>
</tr>
<tr>
<td>6</td>
<td>315</td>
<td>-51.8</td>
<td>350</td>
<td>2.59</td>
</tr>
<tr>
<td>7</td>
<td>269</td>
<td>-54.4</td>
<td>299</td>
<td>2.27</td>
</tr>
<tr>
<td>8</td>
<td>233</td>
<td>-56.3</td>
<td>259</td>
<td>1.02</td>
</tr>
</tbody>
</table>

After the 7th crack forms, the final concrete stress is less than the tensile strength of concrete and the final average crack spacing is therefore about \( s = L / m = 571.4 \text{ mm} \). The final crack width is estimated from Eq. 42:

\[
w = - \left[ (571.4 - 0.667 \times 248)(2.27/7143) - 0.0008 \times 571.4 \right] = 0.33 \text{ mm}.
\]
**Example 3:**

The calculated design average crack width $w$ and average crack spacing $s$ for 6m long members with full end-restraint containing different quantities of 12 mm diameter reinforcing bars and subjected to different levels of shrinkage strain are shown in Table 5. The following material properties were used in the calculations: $f_{ct} = 2.5 \text{ MPa}$; $E_c = 25,000 \text{ MPa}$; $\varphi_{cc} = 2.5$; $E_s = 200000 \text{ MPa}$ and $f_{yk} = 500 \text{ MPa}$. In each case, it is assumed that the early-age cooling strains in the concrete and the reinforcement after the initial heat of hydration were $\varepsilon_{cT} = -0.0002$ and $\varepsilon_{sT} = -0.00024$.

**Table 5:** Effect of shrinkage ($\varepsilon_{cs}$) on the maximum average calculated crack width $w$ (mm) and average crack spacing $s$ (mm) for an end-restrained member ($L = 6000 \text{ mm}$, $d_b = 12 \text{ mm}$, $\varepsilon_{free} = \varepsilon_{cs} + (-0.0002)$)

<table>
<thead>
<tr>
<th>$\varepsilon_{cs}$ ($\times 10^{-6}$)</th>
<th>Reinforcement ratio, $\rho$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.004</td>
<td>0.005</td>
</tr>
<tr>
<td>$w$</td>
<td>0.62</td>
<td>0.48</td>
</tr>
<tr>
<td>$s$</td>
<td>2000</td>
<td>2000</td>
</tr>
<tr>
<td>$w$</td>
<td>0.71</td>
<td>0.50</td>
</tr>
<tr>
<td>$s$</td>
<td>2000</td>
<td>1500</td>
</tr>
<tr>
<td>$w$</td>
<td>0.79</td>
<td>0.52</td>
</tr>
<tr>
<td>$s$</td>
<td>1500</td>
<td>1200</td>
</tr>
</tbody>
</table>

The average crack spacing decreases as the magnitude of the shrinkage strain increases, but the crack width is less affected.

**Table 6** shows the effect of bar diameter on the design average crack width and the average bar spacing for different reinforcement ratios for the following values of the relevant input parameters: $f_{ct} = 2.5 \text{ MPa}$; $E_c = 25,000 \text{ MPa}$; $\varphi_{cc} = 2.5$; $\varepsilon_{cs} = -0.0005$; $\varepsilon_{cT} = -0.0002$; $\varepsilon_{sT} = -0.00024$; $E_s = 200,000 \text{ MPa}$, $f_{yk} = 500 \text{ MPa}$; and $L = 6000 \text{ mm}$.

**Table 6:** Effect of bar diameter on the maximum average calculated crack width $w$ (mm) and average crack spacing $s$ (mm) for an end-restrained member ($L = 6000 \text{ mm}$, $\varepsilon_{free} = -0.0007$).

| $\varnothing$ | Reinforcement ratio, $\rho$-
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(mm)</td>
<td>0.004</td>
<td>0.005</td>
</tr>
<tr>
<td></td>
<td>$w$</td>
<td>$s$</td>
</tr>
<tr>
<td>8</td>
<td>0.52</td>
<td>0.34</td>
</tr>
<tr>
<td>10</td>
<td>0.64</td>
<td>0.44</td>
</tr>
<tr>
<td>12</td>
<td>0.71</td>
<td>0.52</td>
</tr>
<tr>
<td>16</td>
<td>0.93</td>
<td>0.67</td>
</tr>
<tr>
<td>20</td>
<td>1.03</td>
<td>0.85</td>
</tr>
</tbody>
</table>

For a particular reinforcement ratio, both the average crack spacing and the crack width reduce with the bar diameter.
### 4.2 Edge-restraint cracking in slab or wall

Before an edge-restrained wall or slab cracks, the restrained strain is given by Eq. 31 and the deformation resulting from the restraining force $F_t$ over a gauge length $L_o$ is $\varepsilon_r L_o = (\sigma_c / E_{ae}) L_o$. When the maximum concrete stress in the slab or wall $\sigma_c$ exceeds the tensile strength $f_{ct}$, cracking occurs. The crack opens and the concrete stress at the crack drops to zero. Either side of the crack, the concrete stress gradually increases due to bond until at some distance $s_0$ from the crack the concrete stress is unaffected by the crack (as previously illustrated in Figure 17). Cracks form at spacings of between $s_0$ and $2s_0$. If the gauge length $L_o$ is long enough to contain $m$ cracks, the deformation caused by restraint $\varepsilon_r L_o$ may now be expressed as:

$$\varepsilon_r L_o = (\varepsilon_{r,cr} - \varepsilon_{r1}) L_o$$  \hspace{1cm} (56)

where the *residual restrained strain* $\varepsilon_{r1}$ is the sum of the elastic and creep strains caused by the average tensile concrete stress between the cracks and $\varepsilon_{r,cr}$ is the *crack-induced strain* (introduced in Eq. 8). The length $\varepsilon_{r,cr} L_o$ is the sum of the widths of the $m$ cracks within the length $L_o$. That is:

$$\varepsilon_{r1} = \varepsilon_r - \varepsilon_{r,cr} = \sigma_{c,av} / E_{ae}$$  \hspace{1cm} (57)

and

$$\varepsilon_{r,cr} L_o = \sum_{i=1}^{m} w_i$$  \hspace{1cm} (58)

In a member subjected to edge restraint (or one with restraint provided by eccentric reinforcement), the average spacing between cracks depends on the concrete cover, the bond characteristics between the reinforcement and the concrete, the bar diameter, the ratio of reinforcement area to the effective area of the tensile concrete. The maximum crack spacing $s_{r,\text{max}}$ recommended by Eurocode 2 [Ref. 2] is:

$$s_{r,\text{max}} = 3.4 c + (0.425 k_1 k_2 \phi) / \rho_{p,\text{eff}}$$  \hspace{1cm} (59)

where $c$ is the concrete cover to the reinforcement; $\phi$ is the diameter of the reinforcing bars; $\rho_{p,\text{eff}}$ is the ratio of the tensile reinforcement area to the effective area of the tensile concrete ($= A_s / A_{c,\text{eff}}$); $A_s$ is the area of the tensile steel reinforcement crossing the crack and $A_{c,\text{eff}}$ is the effective area of the tensile concrete around the reinforcement. For slabs and walls subjected to edge restraint, $A_{c,\text{eff}}$ may be taken as the gross area of the cross-section. For a member in bending, $A_{c,\text{eff}}$ is the product of the width of the section at the tensile steel level and the depth $h_{c,\text{eff}}$, where $h_{c,\text{eff}}$ is the smaller of $2.5(c + \phi/2)$, $0.5h$ and $(h-x)/3$, where $h$ is the overall depth of the member and $x$ is the depth to the neutral axis. The term $k_1$ in Eq. 59 depends on the bond characteristics of the reinforcement. A value of 0.8 is recommended by Eurocode 2 for high bond bars, but when good bond cannot be guaranteed $k_1$ should be increased to 1.14. This should be applied when considering early-age cracking within the first few days after casting. The coefficient $k_2$ accounts for the distribution of strain. For through thickness cracks caused by edge restraint to shrinkage, $k_2 = 1.0$. For flexural cracking, $k_2 = 0.5$.  

34
The crack width $w$ is determined from:

$$w = s_{r, \text{max}} \epsilon_{c,r} = s_{r, \text{max}} (\epsilon_t - \epsilon_{r1}) \quad (60)$$

The residual strain $\epsilon_{r1}$ given by Eq. 57 and is conservatively approximated by $f_{ct}/E_c$ [Ref. 8] and therefore the crack-induced strain $\epsilon_{c,r}$ may be taken as:

$$\epsilon_{c,r} = \epsilon_t - \epsilon_{r1} = \epsilon_t - f_{ct}/E_c \quad (61)$$

**Example 4:**

For the 200 mm thick wall analysed in Example 1 and shown in Figure 15, the crack spacing and crack width at the base of the wall are to be determined. Each face of the wall is reinforced with 12mm deformed bars running horizontally at 250 mm centres. The cover to the reinforcement is 30 mm and the bond conditions are assumed to be good. Assume that cracking occurs when $f_{ct} = 2.0$ MPa.

For this example, $\phi = 12$ mm, $c = 30$ mm and $A_s = 452$ mm$^2$/m on each face of the wall. The reinforcement ratio is $\rho_{p, \text{eff}} = (2 \times 452)/(200 \times 1000) = 0.00452$ and the maximum crack spacing is determined using Eq. 59:

$$s_{r, \text{max}} = 3.4 \times 30 + (0.425 \times 0.8 \times 1.0 \times 12)/0.00452 = 1005 \text{ mm}$$

Due to the restrained contraction calculated in Example 1, at the base of the wall, $\epsilon_t = 321 \times 10^{-6}$ and, after cracking, Eq. 61 gives:

$$\epsilon_{c,r} = 321 \times 10^{-6} - \frac{2.0}{20000} = 221 \times 10^{-6}$$

and the crack width is calculated using Eq. 60:

$$w = s_{r, \text{max}} \epsilon_{c,r} = 1005 \times 221 \times 10^{-6} = 0.222 \text{ mm}.$$
5. CRACKS CAUSED BY APPLIED LOADS

5.1 Introduction

The control of load-induced cracking in concrete structures may be achieved by limiting the stress in the bonded reinforcement that crosses a crack to some appropriately low value and ensuring that the bonded reinforcement is suitably distributed within the tensile zone. The limit on the tensile steel stress imposed in design should depend on the maximum acceptable crack width. Typical values for maximum acceptable crack widths were given in Figure 1 and Table 1. If the maximum acceptable crack width is increased, the maximum permissible tensile steel stress also increases. The difficulty, of course, is developing a procedure whereby the calculated steel stress provides a reliable indicator of the actual severity of cracking. Building codes usually also specify maximum bar spacing for the bonded reinforcement and maximum concrete cover requirements.

An alternative way to control load-induced cracking is to calculate the maximum crack width and to ensure that the calculated crack width is less than the maximum acceptable crack width. Various deterministic procedures for calculating crack widths are available in the literature. Unfortunately, some of the calculation procedures are overly simplistic and fail to adequately account for the gradual increase in crack widths with time due to shrinkage. In a restrained flexural member, for example, early-age cooling and shrinkage-induced tension in the concrete causes cracking at significantly lower applied loads than are calculated when temperature and shrinkage are not taken into consideration. Shrinkage also causes a gradual widening of flexural cracks and a gradual build-up of tension in the uncracked regions that may lead to additional cracking with time. This has significant implications on the estimates of stiffness in both short-term and long-term deflection calculations. The influence of shrinkage is often not properly considered in some of the widely-used methods for estimating crack widths. As a consequence, excessively wide cracks are a relatively common problem for many reinforced concrete structures.

5.2 Tension Chord Model for Load-induced Cracking in Reinforced Concrete

A model for predicting the final crack width \((w^*)\) in reinforced concrete members based on the Tension Chord Model of Marti et al. [Ref. 12] was proposed by Gilbert [Ref. 13]. A modified version of that model is presented here and is shown to provide good agreement with the measured final spacing and width of cracks in reinforced concrete beams and slabs under sustained loads. The notation associated with the model for a cross-section in bending is shown in Figure 21.

Consider a segment of a singly reinforced beam of rectangular section subjected to an in-service bending moment, \(M_s\), greater than the cracking moment, \(M_{cr}\). After the crack pattern has stabilized, the spacing between the primary cracks is \(s\), as shown in Figure 21a. A typical cross-section between the cracks is shown in Figure 21b and a cross-section at a primary crack is shown in Figure 21c. The cracked beam is idealized as a compression chord of depth \(x = kd\) and
width \( b \) and a cracked tension chord consisting of the tensile reinforcement of area \( A_{st} \) surrounded by an area of tensile concrete \( (A_{ct}) \) as shown in Figure 21d. The centroids of \( A_{st} \) and \( A_{ct} \) are assumed to coincide at a depth \( d \) below the top fibre of the section.

For the sections containing a primary crack (Figure 21c), \( A_{ct} = 0 \) and the depth of the compressive zone \( x = kd \) and the second moment of area about the centroidal axis \( (I_{ct}) \) may be determined from a cracked section analysis using modular ratio theory. For the singly reinforced cross-section shown in Figure 21c:

\[
k = \sqrt{\left(\alpha_e \rho\right)^2 + 2 \alpha_e \rho - \alpha_e \rho}
\]

and

\[
I_{ct} = 0.5bd^3k^2(1 - k/3)
\]

where \( \alpha_e \) is the modular ratio \((= E_s/E_c)\) and \( \rho \) is the reinforcement ratio \((= A_s/bd)\).

Away from the crack, the area of the concrete in the tension chord of Figure 21d \( (A_{ct}) \) is assumed to carry a uniform tensile stress \( (\sigma_{ct}) \) that develops due to the bond stress \( (\tau_b) \) that exists between the tensile steel and the surrounding concrete.

---

**Figure 21:** Cracked reinforced concrete beam and idealized tension chord model [Ref. 13]
For a member in direct tension, the area of the uncracked concrete in the tension chord may be taken as the area of the cross-section. For a member in bending, the area of concrete in the tension chord between the cracks, $A_{ct}$, may be taken as

$$A_{ct} = d_{ct}b^*$$  \hspace{1cm} (64)

where $b^*$ is the width of the section at the level of the centroid of the tensile steel (i.e. at the depth $d$), but not greater than the number of bars in the tension zone $n_{bars}$ multiplied by 12 bar diameters (i.e. 12 $n_{bars}$ $\phi$); and $d_{ct} = 0.5(h - x)$, but should not be taken less than the depth of the area of tensile concrete with a centroid coinciding with the centroid of the tensile steel.

At each crack, the concrete carries no tension and the tensile steel stress is $\sigma_{s1} = T/A_{s}$, where:

$$T = \frac{\alpha_b M_s(1-k)d}{I_{ct}} A_{st}$$  \hspace{1cm} (65)

As the distance $z$ from the crack in the direction of the tension chord increases, the stress in the steel reduces due to the bond shear stress $\tau_b$ between the steel and the surrounding tensile concrete. For reinforced concrete under service loads, where $\sigma_{s1}$ is less than the yield stress $f_{yk}$, Marti et al. [Ref. 12] assumed a rigid-plastic bond shear stress-slip relationship, with $\tau_b = 2.0 f_{ct}$ at all values of slip, where $f_{ct}$ is the direct tensile strength of the concrete. To account for the reduction in bond stress with time due to tensile creep and shrinkage, Gilbert [Ref. 13] took the bond stress to be $\tau_b = 2.0 f_{ct}$ for short-term calculations and $\tau_b = 1.0 f_{ct}$ when the final long-term crack width was to be determined. Experimental observations [Ref. 14 and 15] indicate that $\tau_b$ reduces as the stress in the reinforcement increases and, consequently, the tensile stresses in the concrete between the cracks reduces (i.e. tension stiffening reduces with increasing steel stress). In reality, the magnitude of $\tau_b$ is affected by many factors, including steel stress, concrete cover, bar spacing, transverse reinforcement (stirrups), lateral pressure, compaction of the concrete, size of bar deformations, tensile creep and shrinkage. It is recommended here that in situations where the concrete cover and the clear spacing between the bars are greater than the bar diameter, the bond stress $\tau_b$ may be taken as:

$$\tau_b = \lambda_1 \lambda_2 \lambda_3 f_{ct}$$  \hspace{1cm} (66)

where $\lambda_1$ accounts for the load duration with $\lambda_1 = 1.0$ for short-term calculations and $\lambda_1 = 0.7$ for long-term calculations; $\lambda_2$ is a factor that accounts for the reduction in bond stress as the steel stress $\sigma_{s1}$ (in MPa) increases and may be taken as [Ref. 4]:

$$\lambda_2 = 1.667 - 0.00333 \sigma_{s1}$$  \hspace{1cm} (67)

and $\lambda_3$ is a factor that accounts for the effects on bond stress that have been observed in laboratory tests of the bar diameter $\phi$ and Eq. 68 has been calibrated from observed crack spacings in a wide range of test specimens [Ref. 14].
\[ \lambda_3 = (4.2 - 0.1\phi)(1 - 20\rho_c) \geq 1.5 \]  \hspace{1cm} (68)

The term \( \rho_c \) is the reinforcement ratio of the tension chord \((= A_s/A_c)\) and \( \phi \) is the reinforcing bar diameter in mm.

An elevation of the tension chord is shown in Figure 22a and the stress variations in the concrete and steel in the tension chord are illustrated in Figures 22b and 22c, respectively. Following the approach of Marti et al. [Ref. 12], the concrete and steel tensile stresses in Figures 22b and 22c, where \( 0 < z \leq s/2 \), may be expressed as:

\[
\sigma_{sz} = \frac{T}{A_s} - \frac{4\tau_b z}{\phi} \hspace{1cm} (69)
\]

\[
\sigma_{cz} = \frac{4\tau_b \rho_c z}{\phi} \hspace{1cm} (70)
\]

Mid-way between the cracks, at \( z = s/2 \), the stresses are:

\[
\sigma_{sz} = \frac{T}{A_s} - \frac{2\tau_b s}{\phi} \hspace{1cm} (71)
\]

**Figure 22:** Tension chord - actions and stresses [Ref.13]
\[ \sigma_{c2} = \frac{2 \tau_b \rho_c z}{\phi} \]  

(72)

The maximum crack spacing immediately after loading, \( s = s_{\text{max}} \), occurs when \( \sigma_{c2} = f_{\text{ct}} \) and \( \lambda_1 = 1.0 \) in the determination of \( \tau_b \) in Eq. 66. From Eq. 72:

\[ s_{\text{max}} = \frac{f_{\text{ct}} \phi}{2 \tau_b \rho_c} \leq 2.0 d \]  

(73)

If the spacing between two adjacent cracks just exceeds \( s_{\text{max}} \), the concrete stress mid-way between the cracks will exceed \( f_{\text{ct}} \) and another crack will form between the two existing cracks. It follows that the minimum crack spacing is half the maximum value, i.e. \( s_{\text{min}} = s_{\text{max}}/2 \).

**Instantaneous crack width:** The instantaneous crack width \( \tau_b \) in the fictitious tension chord is the difference between the elongation of the tensile steel over the length \( s \) and the elongation of the concrete between the cracks and is given by:

\[ w_0 = \frac{s}{E_s} \left[ \frac{T}{A_s} - \frac{\tau_b s}{\phi} \right] \]  

(74)

**Maximum final crack width:** Under sustained load, additional cracks occur between widely spaced cracks (usually where \( 0.67s_{\text{max}} < s \leq s_{\text{max}} \)). The additional cracks are due to the combined effect of tensile creep rupture and shrinkage-induced tension. As a consequence, the number of cracks increases and the maximum crack spacing reduces with time. The final maximum crack spacing \( s' \) is only about \( \frac{2}{3} \) of that given by Eq. 73, but the final minimum crack spacing remains about \( \frac{1}{2} \) of the value given by Eq. 73.

Although creep and shrinkage will cause a small increase with time in the resultant tensile force \( T \) in the real beam and a slight reduction in the internal lever arm, this effect is relatively small and is ignored in the tension chord model presented here. The final crack width is the elongation of the steel over the distance between the cracks minus the extension of the concrete caused by \( \sigma_{c2} \) plus the shortening of the concrete between the cracks due to shrinkage \( \varepsilon_{cs} \).

For a final maximum crack spacing of \( s' \), the final maximum crack width \( w' \) at the member soffit is:

\[ w' = \frac{s'}{E_s} \left[ \frac{T}{A_s} - \frac{\tau_b s'}{\phi} \left(1 + \alpha e \rho_c \right) - \varepsilon_{\text{free}} E_s \right] \]  

(75)

where \( s' \) is the maximum crack spacing after all time-dependent cracking has taken place; \( \varepsilon_{\text{free}} \) is the sum of the shrinkage \( \varepsilon_{cs} \) and any thermal strain in the concrete \( \varepsilon_{CT} \) (and \( \varepsilon_{\text{free}} \) is a negative value); \( \alpha e = E_s /E_c \) = the effective modular ratio; \( E_c = E_c/(1 + \varphi c) \) = the **effective modulus** of concrete; \( E_c \) and \( E_s \) are the elastic moduli of the concrete and steel, respectively; \( \tau_b \) is determined from Eq. 66 with \( \lambda_1 = 0.7 \) and \( \varphi c \) is the creep coefficient of the concrete.
A good estimate of the final maximum crack width is given by Eq. 75, when \( s^* \) is taken as \( \frac{2}{3} \) the initial value of \( s_{\max} \) given by Eq. 73 (with \( \lambda_1 = 1.0 \) in the determination of \( \tau_b \)). That is

\[
s^* = \frac{f_{ct} \phi}{3 \tau_b \rho_c} \leq 2.0d
\]  

(76)

By rearranging Eq. 75, the steel stress on a cracked section corresponding to a particular maximum final crack width \( w^* \) is given by:

\[
f_{at} = \frac{w^* E_s}{s^*} + \frac{\tau_b s^*}{\phi} (1 + \alpha_e \rho_c) + \epsilon_{free} E_s
\]  

(77)

By substituting \( \tau_b \) (from Eqs. 66) and \( s^* \) (from Eq. 76) into Eq. 77 and by selecting a maximum desired crack width in a particular structure \( w^* \), the maximum permissible tensile steel stress can be determined.

**Example 5:**

A 150 mm thick simply-supported one-way slab located inside a building is to be considered. With appropriate regard for durability, the concrete strength is selected to be \( f'_{c} = 32 \) MPa and the cover to the tensile reinforcement is 20 mm. The final shrinkage strain is taken to be \( \epsilon_{free} = \epsilon_{cs} = -0.0006 \) (with \( \epsilon_{ct} = 0 \)). Other relevant material properties are \( E_c = 28,600 \) MPa; \( \alpha_c = E_s/E_c = 7.0; \phi_c = 2.5; f_{ct} = 3.0 \) MPa and \( E_s = 200,000 \) MPa. The effective modulus is therefore \( E_e = E_c/(1+\phi_c) = 8,170 \) MPa and the effective modular ratio \( \alpha_{ee} = E_s/E_e = 24.5 \). The tensile face of the slab is to be exposed and the maximum final crack width is to be limited to \( w^* = 0.3 \) mm.

After completing the design for strength and deflection control, the required minimum area of tensile steel is 650 mm²/m. Under the full service loads, the maximum in-service sustained moment at mid-span is 20.0 kNm/m. The bar diameter and bar spacing must be determined so that the requirements for crack control are also satisfied.

**Case 1** - Try 10 mm bars at 120 mm centres:

The area of tensile steel is \( A_s = 655 \) mm²/m at \( d = 125 \) mm and, referring to Figure 19, an elastic analysis of the cracked section gives \( x = kd = 29.6 \) mm and \( l_{cr} = 50.3 \times 10^6 \) mm⁴ (from Eqs. 62 and 63). The maximum in-service tensile steel stress on the fully-cracked section at mid-span is calculated using Eq. 65 as: \( \sigma_{s1} = T/A_s = 265 \) MPa.

The area of concrete in the tension chord is \( A_{ct} = 60,200 \) mm² (Eq. 64) and the reinforcement ratio of the tension chord is \( \rho_c = A_s/A_{ct} = 0.0109 \). From Eqs. 67 and 68, \( \lambda_2 = 0.783 \) and \( \lambda_3 = 2.50 \) and, from Eq. 66, \( \tau_b = 5.88 \) MPa for short-term calculations (\( \lambda_1 = 1.0 \)) and \( \tau_b = 4.11 \) MPa for long-term calculations (\( \lambda_1 = 0.7 \)). The maximum final crack spacing \( s^* \) is obtained from Eq. 76:

\[
s^* = \frac{3.0 \times 10^6}{3 \times 5.88 \times 0.0109} = 156 \text{ mm} \leq 2.0d
\]
The maximum permissible steel stress required for crack control is obtained from Eq. 77:

\[
f_{st} = \frac{0.3 \times 200000}{156} + \frac{4.11 \times 156}{10} (1 + 24.5 \times 0.0109) + (-0.0006 \times 200000) = 346 \text{ MPa.}
\]

The actual stress at the crack \( \sigma_{s1} = 265 \text{ MPa} \) is much less than \( f_{st} = 346 \text{ MPa} \) and, therefore, cracking is easily controlled.

**Case 2** - Use 12 mm bars at 170 mm centres:

The area of tensile steel is \( A_{st} = 665 \text{ mm}^2/\text{m} \) at \( d = 124 \text{ mm} \). For this section, \( x = kd = 29.6 \text{ mm} \) and \( I_{cr} = 50.1 \times 10^6 \text{ mm}^4 \). The maximum in-service tensile steel stress on the fully-cracked section at mid-span is \( \sigma_{s1} = T/A_s = 264 \text{ MPa.} \)

The area of concrete in the tension chord is \( A_{ct} = 50,980 \text{ mm}^2 \) (Eq. 64) and \( \rho_c = A_s/A_{ct} = 0.0130 \). Now \( \lambda_2 = 0.787 \) and \( \lambda_3 = 2.22 \) and \( \tau_b = 5.24 \text{ MPa} \) for short-term calculations (with \( \lambda_1 = 1.0 \)) and \( \tau_b = 3.67 \text{ MPa} \) for long-term calculations (with \( \lambda_1 = 0.7 \)). The maximum final crack spacing is \( s^* = 176 \text{ mm} \). From Eq. 77, the maximum permissible steel stress required for crack control is \( f_{st} = 292 \text{ MPa} \) and this is also significantly greater than the actual maximum stress at the crack \( \sigma_{s1} = 264 \text{ MPa} \).

Therefore, the final maximum crack width will be less than the maximum permissible value of 0.3 mm.

**Case 3** - Use 16 mm bars at 300 mm centres:

The area of tensile steel is \( A_s = 670 \text{ mm}^2/\text{m} \) at \( d = 122 \text{ mm} \). For this section, \( x = kd = 29.4 \text{ mm} \) and \( I_{cr} = 48.6 \times 10^6 \text{ mm}^4 \). The maximum in-service tensile steel stress on the fully-cracked section at mid-span is \( \sigma_{s1} = T/A_s = 266 \text{ MPa.} \) The area of concrete in the tension chord is \( A_{ct} = 38,580 \text{ mm}^2 \) and \( \rho_c = A_s/A_{ct} = 0.0174 \). Now \( \lambda_2 = 0.780 \) and \( \lambda_3 = 1.70 \) and \( \tau_b = 3.97 \text{ MPa} \) for short-term calculations (with \( \lambda_1 = 1.0 \)) and \( \tau_b = 2.78 \text{ MPa} \) for long-term calculations (with \( \lambda_1 = 0.7 \)). The maximum final crack spacing is \( s^* = 232 \text{ mm} \). The maximum permissible steel stress required for crack control is \( f_{st} = 196 \text{ MPa} \) (from Eq. 77), which is significantly less than the actual steel stress due to the sustained moment (\( \sigma_{s1} = 266 \text{ MPa} \)). Therefore, crack control is not adequate and the maximum final crack width will exceed 0.3 mm.

### 5.3 Comparison with test data

The maximum final crack widths determined using Eq. 75 are compared with the measured maximum final crack widths for twelve prismatic, one-way, singly reinforced concrete specimens (6 beams and 6 slabs) that were tested under sustained service loads for periods in excess of 400 days by Gilbert and Nejadi [Ref. 14]. The specimens were simply-supported over a span of 3.5 m with cross-sections shown in Figure 23. All specimens were cast from the same batch of concrete and moist cured prior to first loading at age 14 days. Details of each test specimen are given in Table 7.
The time-dependent variations of crack spacing and crack width, as well as the crack locations and heights, were measured in each specimen throughout the test. The measured elastic modulus and compressive strength of the concrete at the age of first loading were $E_c = 22,820$ MPa and $f_c = 18.3$ MPa, whilst the measured creep coefficient and shrinkage strain associated with the 400 day period of sustained loading were $\varphi_{cc} = 1.71$ (measured on 300 mm cylinders) and $\varepsilon_{cs} = -0.000825$ (measured on standard 75 mm prisms).

Two identical specimens “a” and “b” were constructed for each combination of parameters as indicated in Table 7 and 8, with the “a” specimens loaded more heavily than the “b” specimens. The “a” specimens were subjected to a constant sustained load sufficient to cause a maximum moment at mid-span of between 40 and 50% of the calculated ultimate moment and the “b” specimens were subjected to a constant sustained mid-span moment of between 25 and 40% of the calculated ultimate moment.

The loads on all specimens were sufficient to cause primary cracks to develop in the region of maximum moment at first loading. In Table 8, the sustained in-service moment at mid-span, $M_{sus}$, is presented, together with the stress in the tensile steel at mid-span, $\sigma_{s1}$, due to $M_{sus}$ (calculated on the basis of a fully cracked section); the calculated ultimate flexural strength, $M_u$ (assuming a characteristic yield stress of the reinforcing steel of 500 MPa); the ratio $M_{sus}/M_u$; and the cracking moment, $M_{cr}$, (calculated assuming a tensile strength of concrete of $0.6\sqrt{f_c(t)}$, where $f_c(t)$ is the measured compressive strength at the time of loading in MPa).
At first loading, a regular pattern of primary cracks developed in each test specimen. With time, the cracks gradually increased in width and additional cracks developed between some of the primary cracks. Thus, the average crack spacing reduced with time. The ratio of final to initial crack spacing ranged from 0.57 to 0.85, with an average value of 0.70. Crack widths increased rapidly in the first few weeks after loading, but the rate of increase slowed significantly after about 2 months. For all specimens, there was little change in the maximum crack width after about 200 days under load. Typical calculations for the maximum crack width are provided here for Beam B1-a

**Beam B1-a:**

Relevant dimensions and properties are: \( b = 250 \text{ mm}, h = 348 \text{ mm}, d = 300 \text{ mm}, d_h = 16 \text{ mm}, A_s = 400 \text{ mm}^2, E_c = 22,820 \text{ MPa} \) and \( \alpha_e = E_s / E_c = 8.76 \). A cracked section analysis gives \( kd = 78.8 \text{ mm} \) and \( I_{cr} = 212 \times 10^6 \text{ mm}^4 \). The applied moment at mid-span is \( M_{sus} = 24.9 \text{ kNm} \) and the stress in the tensile steel on the cracked section is

\[
\sigma_s = \frac{M_{sus} (d - kd)}{I_{cr}} = \frac{8.76 \times 24.9 \times 10^6 \times (300 - 78.8)}{212 \times 10^6} = 227.4 \text{ MPa}
\]

The measured final creep coefficient was \( \phi_{cc} = 1.71 \) and, therefore \( E_e = 8,420 \text{ MPa} \) and \( \alpha_{ee} = 23.8 \). The final shrinkage strain measured on a slab specimen 160 mm deep and 400 mm wide (with hypothetical thickness \( t_h = 114.3 \text{ mm} \) was \( -0.000825 \). For this beam specimen, with \( t_h = 145.5 \text{ mm} \), the estimated shrinkage strain is \( \varepsilon_{free} = -0.000746 \).

With \( d_{ct} = 0.5(D - kd) = 0.5 \times (348 - 78.8) = 134.6 \text{ mm} \), the area of concrete in the tension chord is given by Eq. 64 as \( A_{ct} = 134.6 \times 250 = 33,650 \text{ mm}^2 \) and the reinforcement ratio of the tension chord \( \rho_{tc} = A_s / A_{ct} = 0.0119 \).

To determine the maximum crack spacing for short term calculations immediately after cracking, \( \lambda_1 = 1.0 \) and, from Eqs. 67 and 68:

\[
\lambda_2 = 1.667 - 0.00333 \times 227.4 = 0.909 \quad \text{and} \quad \lambda_3 = (4.2 - 0.1 \times 16)(1 - 20 \times 0.0119) = 1.982
\]

and with \( f_{ct} = 0.6 \sqrt{f_c(t)} = 2.57 \text{ MPa} \), the instantaneous bond stress is obtained from Eq. 66 as:

\[
\tau_b = 1.0 \times 0.909 \times 1.982 \times 2.57 = 4.622 \text{ MPa}
\]

\[
\alpha_e M_{sus} (d - kd)
\]

\[
\sigma_s = \frac{M_{sus} (d - kd)}{I_{cr}} = \frac{8.76 \times 24.9 \times 10^6 \times (300 - 78.8)}{212 \times 10^6} = 227.4 \text{ MPa}
\]

![Table 8: Moments and steel stresses in the test specimens [Ref. 14]](image)

<table>
<thead>
<tr>
<th>Beam</th>
<th>( M_{cr} ) kNm</th>
<th>( M_{sus} ) kNm</th>
<th>( \sigma_s ) MPa</th>
<th>( M_u ) kNm</th>
<th>( M_{sus}/M_u ) (%)</th>
<th>Slab</th>
<th>( M_{cr} ) kNm</th>
<th>( M_{sus} ) kNm</th>
<th>( \sigma_s ) MPa</th>
<th>( M_u ) kNm</th>
<th>( M_{sus}/M_u ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>B1-a</td>
<td>14.0</td>
<td>24.9</td>
<td>227</td>
<td>56.2</td>
<td>44.3</td>
<td>S1-a</td>
<td>4.65</td>
<td>6.81</td>
<td>252</td>
<td>13.9</td>
<td>49.0</td>
</tr>
<tr>
<td>B1-b</td>
<td>14.0</td>
<td>17.0</td>
<td>155</td>
<td>56.2</td>
<td>30.2</td>
<td>S1-b</td>
<td>4.65</td>
<td>5.28</td>
<td>195</td>
<td>13.9</td>
<td>38.0</td>
</tr>
<tr>
<td>B2-a</td>
<td>13.1</td>
<td>24.8</td>
<td>226</td>
<td>56.2</td>
<td>44.1</td>
<td>S2-a</td>
<td>4.75</td>
<td>9.87</td>
<td>247</td>
<td>20.3</td>
<td>48.6</td>
</tr>
<tr>
<td>B2-b</td>
<td>13.1</td>
<td>16.8</td>
<td>153</td>
<td>56.2</td>
<td>29.8</td>
<td>S2-b</td>
<td>4.75</td>
<td>6.81</td>
<td>171</td>
<td>20.3</td>
<td>33.6</td>
</tr>
<tr>
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<td>34.6</td>
<td>214</td>
<td>81.5</td>
<td>42.4</td>
<td>S3-a</td>
<td>4.86</td>
<td>11.4</td>
<td>216</td>
<td>26.4</td>
<td>43.0</td>
</tr>
<tr>
<td>B3-b</td>
<td>13.7</td>
<td>20.8</td>
<td>129</td>
<td>81.5</td>
<td>25.5</td>
<td>S3-b</td>
<td>4.86</td>
<td>8.34</td>
<td>159</td>
<td>26.4</td>
<td>31.6</td>
</tr>
</tbody>
</table>
and the maximum crack spacing immediately after loading is given by Eq. 73:

\[ s_{\text{max}} = \frac{2.57 \times 16}{2 \times 4.622 \times 0.0119} = 373.7 \text{ mm} \]

For the calculation of the maximum final crack width, the maximum crack spacing \( s^* \) is taken as \( \frac{2}{3} \) of the instantaneous value and therefore \( s^* = \frac{2}{3} \times 373.7 = 249.2 \text{ mm} \) (Eq. 76). From Eq. 66, for long-term calculations, \( \tau = 0.7 \times 0.909 \times 1.982 \times 2.57 = 3.235 \text{ MPa} \). The maximum final (long-term) crack width at the soffit of the beam specimen B1-a is obtained from Eq. 62:

\[
\begin{align*}
  w^* &= \frac{249.2}{200000} \left[ 227.4 - \frac{3.235 \times 249.2}{16} \left( 1 + 23.8 \times 0.0119 \right) - (-0.000746 \times 200000) \right] \\
  &= 0.389 \text{ mm}
\end{align*}
\]

The measured maximum final crack width on Beam B1-a after 400 days under load was 0.38 mm which is in good agreement. The measured and calculated maximum final crack widths for all twelve test specimens are compared in Table 9. The mean of the ratios of predicted to measured crack widths for the six beam specimens is 1.155, with a coefficient of variation of 11.75%, whilst for the six slab specimens the mean is 1.070, with a coefficient of variation of 12.31%. The agreement between the calculated and measured maximum final crack widths for this set of test data is good.

<table>
<thead>
<tr>
<th>Specimen</th>
<th>B1-a</th>
<th>B1-b</th>
<th>B2-a</th>
<th>B2-b</th>
<th>B3-a</th>
<th>B3-b</th>
<th>S1-a</th>
<th>S1-b</th>
<th>S2-a</th>
<th>S2-b</th>
<th>S3-a</th>
<th>S3-b</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured, ( w_{\text{max}} )</td>
<td>0.38</td>
<td>0.18</td>
<td>0.36</td>
<td>0.18</td>
<td>0.28</td>
<td>0.13</td>
<td>0.25</td>
<td>0.20</td>
<td>0.23</td>
<td>0.18</td>
<td>0.20</td>
<td>0.15</td>
</tr>
<tr>
<td>Predicted, ( w^* )</td>
<td>0.389</td>
<td>0.236</td>
<td>0.377</td>
<td>0.228</td>
<td>0.288</td>
<td>0.163</td>
<td>0.307</td>
<td>0.212</td>
<td>0.283</td>
<td>0.170</td>
<td>0.205</td>
<td>0.140</td>
</tr>
<tr>
<td>( w^*/w_{\text{max}} )</td>
<td>1.023</td>
<td>1.313</td>
<td>1.046</td>
<td>1.269</td>
<td>1.029</td>
<td>1.251</td>
<td>1.228</td>
<td>1.062</td>
<td>1.229</td>
<td>0.944</td>
<td>1.024</td>
<td>0.935</td>
</tr>
</tbody>
</table>

6. CONCLUDING REMARKS

Rational procedures have been proposed for determining the degree of restraint and the control of cracking caused by early-age thermal contraction and shrinkage of concrete and by applied loads. Restraining forces and concrete tensile stresses caused by restraint to shrinkage strains in a variety of situations have been considered, including concrete members subjected to temperature differentials, reinforced concrete members containing embedded reinforcement, reinforced concrete beams, slabs and walls subjected to end restraint and reinforced walls and slabs subjected to edge-restraint and reinforced concrete members subjected to axial tension and bending. Calculation methods of the width and spacing of cracks caused by any one or any combination of these actions are outlined and illustrated by several worked examples.

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