LATERAL - DISTORTIONAL BUCKLING OF CONTINUOUSLY RESTRAINED COLUMNS

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A rational model for predicting the elastic buckling load of thin-walled I-section columns restrained fully against translation and elastically against twist at one flange is presented. The energy method used results in a third order eigenproblem. The numerical method is illustrated by considering the elastic buckling of doubly-symmetric and monosymmetric columns with various degrees of elastic twist restraint. It is shown that the buckling mode is of a lateral-distortional type. The accuracy of the U-frame buckling model used in through girders for predicting the lateral-distortional buckling load is assessed in the study.
INTRODUCTION

Members subjected to axial compression, in the absence of local buckling, are generally considered to fail due to instability effects in (i) a flexural mode; (ii) a torsional mode or (iii) a flexural-torsional mode. For typical doubly-symmetric I-section members, the buckling mode for an unrestrained column is generally its flexural mode, which is characterised by buckling about its minor axis [1]. Monosymmetric columns, however, may buckle in a flexural-torsional mode as the shear centre and centroid do not coincide [2].

Many columns are deployed in situations where they are restrained along their length. For example, a column in a portal frame structure may be restrained by girts and sheeting attached to one flange, and if the girt spacing is close and the column is long, the restraining action may be considered as ‘smeared’ and therefore continuous. It is the elastic buckling of such continuously restrained columns that is the subject of this paper.

The failure modes of restrained columns, considered in such standard works as Horne and Ajmani [3] and Trahair [2], treat the cross-section as rigid, so that classical Vlasov-type flexural, torsional and flexural-torsional buckling theories may be applied. It is shown in this paper that when the assumption of a rigid cross-section is relaxed, the restrained column will buckle in a so-called lateral-distortional buckling mode [4] in which the web of the column distorts in the plane of its cross-section. Lateral-distortional buckling modes are intermediate between flexural-torsional and local buckling, and for unrestrained I-section members have generally been shown to be of minor significance. However, for I-section members with partial restraint [5] or continuous restraint [6,7], lateral-distortional buckling is of significance, and the flexural-torsional assumption of a rigid cross-section is incorrect. For example, if a column is fully restrained against translation and twist of one of its flanges, Vlasov theory predicts an infinite elastic buckling load. This of course is not the case, and the unrestrained compression flange
buckles sideways and twists, accompanied by distortion of the web, in the lateral-distortional buckling mode depicted in Fig. 1.

In portal frame structures, the girts and sheeting provide restraint against translation and twist of the flange of the column to which they are connected. The girt/sheeting combination may be expected to provide close to fully effective translational restraint in the plane of the sheeting, but the restraint against twist will generally be elastic with a stiffness per unit length of $2EI_s/L_c$, where $I_s$ is the second moment of area per unit length of the girt/sheeting combination for flexure, and $L_c$ is the length between adjacent columns. This behaviour is depicted in Fig. 2. Usual design practice is to specify fly bracing to the unrestrained compression flange [8] in order to stabilise it against buckling.

This paper considers the elastic lateral-distortional buckling of a monosymmetric column with one flange fully restrained against translation, but with the ability of this flange to be restrained elastically against twist. The other flange of the I-section column is considered to be restrained only by the stiffness of the web. An energy method of analysis is used to develop an eigenproblem of order three, and some numerical solutions are presented. In addition, the so-called U-frame buckling model is introduced, and its accuracy in describing this lateral-distortional buckling problem is assessed.

ENERGY-BASED DISTORTIONAL BUCKLING SOLUTION

Consider a pin-ended column of monosymmetric I-section, with the top (T) flange restrained fully against translation and elastically against twist, but with the bottom (B) flange free to deflect and twist. The buckled cross-section of this column is shown in Fig. 3. The top and bottom flanges twist $\phi_T$ and $\phi_B$, and the bottom flange displaces $u_B$. It is assumed that these deformations vary sinusoidally along the member, so that the buckling deformations can be written as
\[
\begin{align*}
\begin{pmatrix}
u_B \\
\phi_T \\
\phi_B
\end{pmatrix} &= 
\begin{pmatrix}
q_1 \\
q_2 \\
q_3
\end{pmatrix} \sin n \pi \xi \\
\end{align*}
\]

where $\xi = z/L$, $q_1$, $q_2$ and $q_3$ are the maximum amplitudes of the buckling displacements and $n$ is a positive integer representing the number of harmonics into which the column buckles.

A cubic equation is used to describe the deformations in the web during buckling, so that

\[
u_w = h \left( \alpha_1 + \alpha_2 \eta + \alpha_3 \eta^2 + \alpha_4 \eta^3 \right) \sin n \pi \xi
\]

where $\eta = y/h$ and $\alpha_1, ..., \alpha_4$ are dimensionless undetermined polynomial coefficients. If it assumed that a right angle exists at the flange-web junctions, then the following compatibility conditions must be met:

\[0 = (u_w)_{\xi=1/2, n=-1/2} \quad (3a)\]

\[u_B = (u_w)_{\xi=1/2, n=1/2} \quad (3b)\]

\[\phi_T = (\partial u_w / \partial y)_{\xi=1/2, n=-1/2} \quad (3c)\]

\[\phi_B = (\partial u_w / \partial y)_{\xi=1/2, n=1/2} \quad (3d)\]

Making this substitution, and solving for the undetermined coefficients produces
\[
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4
\end{pmatrix} = 
\begin{bmatrix}
1/2h & -1/8 & 1/8 \\
3/2h & 1/4 & 1/4 \\
0 & 1/2 & -1/2 \\
-2/h & -1 & -1
\end{bmatrix}
\begin{pmatrix}
q_1 \\
q_2 \\
q_3
\end{pmatrix}
\tag{4}
\]

During buckling, the monosymmetric column displaces and twists, and strain energy \( U \) is stored in the member. The first variation of this strain energy may be written as

\[
\delta U = \delta U_F + \delta U_W + \delta U_R
\tag{5}
\]

where \( U_F \) is the strain energy stored in the flanges, \( U_W \) is the strain energy stored in the web and \( U_R \) is the strain energy stored in the elastic twist restraint at the top flange. By assuming that the flanges buckle as rigid bodies, the contribution \( \delta U_F \) can be written as [9]

\[
\delta U_F = \int_0^L \{\delta \varepsilon_F\}^T [D_F] \{\varepsilon_F\} dz
\tag{6}
\]

where \([D_F]\) is a property matrix given by

\[
[D_F] = 
\begin{bmatrix}
(EI_y)_B & 0 & 0 \\
0 & (GJ)_T & 0 \\
0 & 0 & (GJ)_B
\end{bmatrix}
\tag{7}
\]

in which \( EI_y \) is the minor axis flexural rigidity of a flange and \( GJ \) is its torsional rigidity, and the generalised strain vector \( \{\varepsilon_F\} \) is

\[
\{\varepsilon_F\} = \left( \partial^2 u_B / \partial z^2, \partial \Phi_T / \partial z, \partial \phi_B / \partial z \right)^T
\tag{8}
\]
and so

$$\delta U_F = \{\delta q\}^T [k_F]\{q\}$$  \hspace{1cm} (9)$$

where \([k_F]\) is the flange stiffness matrix.

The strain energy \(U_W\) stored in the web during buckling can be obtained by assuming that the flexible web buckles as an isotropic plate \([10]\). The first variation of the strain energy may be written as \([9]\)

$$\delta U_W = \int_0^{l} \int_{-h/2}^{h/2} \{\delta \varepsilon_W\}^T [D_W]\{\varepsilon_W\} \, dy \, dz$$  \hspace{1cm} (10)$$

where \([D_W]\) is the property matrix given by

$$[D_W] = \frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$  \hspace{1cm} (11)$$

and the web generalised strain vector \(\{\varepsilon_W\}\) is

$$\{\varepsilon_W\} = \begin{bmatrix} \frac{\partial^2 u_w}{\partial y^2}, \frac{\partial^2 u_w}{\partial z^2}, -2\frac{\partial^2 u_w}{\partial y \partial z} \end{bmatrix}^T$$  \hspace{1cm} (12)$$

Hence from Eqs. 10, 11, 12, 2 and 4

$$\delta U_W = \{\delta q\}^T [k_W]\{q\}$$  \hspace{1cm} (13)$$
where \([k_W]\) is the web stiffness matrix.

If \(k_z\) is the continuous twist restraint of the restrained flange per unit length, then the first variation of \(U_R\) is [2]

\[
\delta U_R = k_z \int_0^L \delta \phi \cdot \phi \, dz
\]  

(14)

which may be expressed from Eq. 1 as

\[
\delta U_R = [\delta \phi]^T [k_R] [\phi]
\]  

(15)

where \([k_R]\) is the restraint stiffness matrix.

When the column buckles, the applied stresses \(\lambda \sigma\) undergo displacements which result in work \(V\) being done, where \(\lambda\) is the buckling load factor. The first variation of this work may be written as

\[
\delta V = \delta V_F + \delta V_W
\]  

(16)

where \(\delta V_F\) is the work done by the stresses in the flanges and \(\delta V_W\) is the work done by the stresses in the web. The contribution \(\delta V_F\) may be written as [9]

\[
\delta V_F = \sigma A_F \int_0^L \left[ \left( \frac{\partial \delta u_0}{\partial z} \right) \left( \frac{\partial u_0}{\partial z} \right) + \left( \frac{\partial \delta v_{1}}{\partial z} \right) \left( \frac{\partial v_{1}}{\partial z} \right) + \left( \frac{\partial \delta v_{2}}{\partial z} \right) \left( \frac{\partial v_{2}}{\partial z} \right) \right] \, dz
\]  

(17)

where \(A_F\) is the area of the flanges and where
\[ \nu_T = x \phi_T \]  

\[ \nu_B = x \phi_B \]  

Hence on substituting for the displacements,

\[ \delta V_F = [\delta q]^T \lambda [g_F] [q] \]  

(19)

where \([g_F]\) is the flange stability matrix.

Similarly, the first variation of the work done as the stresses displace in the web is [9]

\[ \delta V_w = \sigma A_w \int_0^L \left( \frac{\partial \phi_{z_1}}{\partial z} \right) \left( \frac{\partial \phi_{z_2}}{\partial z} \right) dz \]  

(20)

in which \(A_w\) is the area of the web. Hence,

\[ \delta V_w = [\delta q]^T \lambda [g_w] [q] \]  

(21)

For equilibrium of the buckled configuration, the first variation of the total potential \(\Pi\) given by

\[ \delta \Pi = \delta U - \delta V \]  

(22)

must vanish. Hence from Eqs. 9, 13, 19 and 21

\[ [\delta q]^T ([k] - \lambda [g]) [q] = 0 \]  

(23)
where

\[
[k] = [k_F] + [k_W] + [k_R]
\]  \hspace{1cm} (24)

and

\[
[g] = [g_F] + [g_W]
\]  \hspace{1cm} (25)

are the 3x3 stiffness and stability matrices respectively of the restrained column. Since the variations \(\{\delta q\}\) in Eq. 23 are arbitrary,

\[
([k] - \lambda [g])\{q\} = \{0\}
\]  \hspace{1cm} (26)

Equation 26 is a standard eigenproblem which may be solved for the buckling load factor or eigenvalue \(\lambda\) and the buckled shape or eigenvector \(\{q\}\). Although only requiring the solution of a cubic equation, the coefficients in \([k]\) and \([g]\) are not straightforward, and so a standard eigensolver was invoked to solve Eq. 26.

U-FRAME MODEL

The so-called U-frame model is often used for the buckling analysis of through girders. For this, a pair of equal and opposite unit forces per unit length are applied at the top flange level, as shown in Fig. 4a, and the resulting displacements \(\Delta\) are obtained. The top or compression flange is then considered as a strut restrained elastically against translation by a web and web/flange combination of stiffness \(k_I = 1/\Delta\) per unit length [11].
In the analysis of the problem consider a strut subjected to an axial load \( N \) and restrained continuously by a translational restraint \( k_t \) per unit length, as in Fig. 4b. If the strut displaces \( u \) during buckling, then it is assumed that

\[
u = q \sin n\pi \xi
\]  

(27)

The first variation of the strain energy stored is

\[
\delta U = EI_f \int_0^L \delta q^* q^* \, dz + k_t \int_0^L \delta u \cdot u \, dz
\]  

(28)

where \( EI_f \) is the flexural rigidity of the restrained flange, and where primes denote differentiation with respect to \( z \). Substituting Eq. 28 into Eq. 27 gives

\[
\delta U = \delta q \left( \frac{n^2 \pi^2 EI_f}{2L} + \frac{k_t L}{2} \right) q
\]  

(29)

while the first variation of the work done is

\[
\delta V = N \int_0^L \delta u' \cdot u' \, dz
\]  

(30)

so that

\[
\delta V = \delta q \left( \frac{Nn^2 \pi^2}{2L} \right) q
\]  

(31)

Invoking the principle of virtual work, viz. \( \delta U - \delta V = 0 \), produces
\[ \delta q \left( \frac{EI_f n^2 \pi^4}{2L^2} + \frac{k_t L}{2} - \frac{N n^2 \pi^2}{2L} \right) q = 0 \] (32)

Again, noting \( \delta q \) is arbitrary and that \( q \) is nonzero produces the critical load

\[ N_{cr} = \frac{n^2 \pi^2 EI_f}{L^2} + \frac{k_t L^2}{n^2 \pi^2} \] (33)

Plotting Eq. 33 against \( L \) produces a series of garland curves depending on the harmonic number \( n \). The lower bound for these curves represents the elastic buckling load in the compression flange that is restrained by the web. Treating the web as a cantilever of length \( h \), then

\[ \Delta_1 = \frac{h^3}{3EI_w} \] (34)

where \( I_w = r^3 / 12 \). The unit force also produces a moment \( L h \) and hence a rotation \( h / k_z \) at the twist restraint. This results in an additional deformation

\[ \Delta_2 = h^2 / k_z \] (35)

Finally,

\[ \Delta = \Delta_1 + \Delta_2 \] (36)

and the translational restraint in Eq. 33 is

\[ k_t = 1 / \Delta \] (37)
BUCKLING OF DOUBLY-SYMMETRIC COLUMNS

Doubly symmetric I-section columns without continuous restraint usually buckle about their minor axis in a flexural mode, although in theory torsional buckling may occur. However, this is not the case when continuous restraints are present. Figures 5 and 6 show the buckling load $N_{cr}$, normalised with respect to the Euler load $N_E$, as a function of the dimensionless length $L/h$ and the torsional restraint parameter $\alpha_z$.

$$\alpha_z = \frac{k_z}{\pi^2 GJ / L^2}$$  \hfill (38)

Figure 5 is plotted with $h/t = 100$, while Fig. 6 is for a thinner web with $h/t = 200$, the remaining geometry being as indicated in the figures.

The column with the thicker web (Fig. 5) buckles in one harmonic ($n = 1$) for values of $\alpha_z = 100$ and 1000 for $L/h$ less than about 17, but buckles in two harmonics ($n = 2$) for values of $L/h$ greater than this. Owing to the restraint, the elastic critical loads are all greater than the Euler load over the range of lengths considered, but the ratio $N_{cr} / N_E$ is much less for the thinner web (Fig. 6) due to the distortion of the web. In order to demonstrate this distortion, Fig. 7 plots the normalised buckling mode $\{q\}$ in Eq. 26 at the maximum buckling amplitude ($\zeta = 1/2n$). It can be seen that when $\alpha_z = 0$ and there is only translational restraint, the column buckles in a flexural-torsional mode. However, as the twist restraint $\alpha_z$ increases, the effects of web distortion increase and the buckling mode is lateral-distortional. For a given value of the twist restraint parameter, the distortion is greater for the thinner web as the twist of the restrained flange is less, although for both web thicknesses the relative twist of the unrestrained flange is quite similar. When $\alpha_z$ reaches 1000, the flanges for both columns are effectively fully restrained against twist.
Figures 5 and 6 also show the predictions of the U-frame model for distortional buckling. Of course, for $\alpha_z = 0$, the flexibilities $\Delta_2$ (Eq. 35) and $\Delta$ (Eq. 36) are infinite, and so $k_t = 0$ and the buckling load becomes the Euler load $N_E$ for this value of the twist restraint parameter. While this prediction is very conservative, much of the conservatism is lost for larger values of $\alpha_z$, particularly for the thinner web (Fig. 6). It is noteworthy that for $h/t = 100$ (Fig. 5) the U-frame model also predicts a cross-over point for buckling modes, but this occurs near $L/h = 20$, whereas the numerical model predicts the mode transition near $L/h = 17$.

BUCKLING OF MONOSYMMETRIC COLUMNS

In fabricated welded columns, it may be advantageous to economise on material by making one flange smaller than the other. It is therefore instructive to consider the buckling of restrained monosymmetric I-section columns.

Figure 8 plots the non-dimensional critical load $N_{cr}/N_E$ when the unrestrained flange has twice the width of the elastically restrained flange, while Fig. 9 plots $N_{cr}/N_E$ when the restrained flange has twice the width of the unrestrained flange. Curves are given for values of $\alpha_z$ of 0, 10, 100 and 1000, and both columns have a web slenderness of $h/t = 100$. It is evident from the figures that when the smaller flange is the unrestrained flange, the elastic critical loads are lower. For all values of $\alpha_z$ except zero, there is a switch between one ($n = 1$) and two ($n = 2$) buckling harmonics in Fig. 8 near $L/h = 19$. On the other hand, when the smaller flange is unrestrained as in Fig. 9, there are up to four buckling harmonics ($n = 1, \ldots, 4$) represented on the graph.

The predictions of the U-frame approach are also plotted in Figs. 8 and 9. It can be seen that for a monosymmetric column, the U-frame approximation is quite unconservative compared with the numerical solutions. Finally, Fig. 10 shows the normalised buckled shapes $\{q\}$ obtained from the eigensolver. The figure shows that the
degree of web distortion is fairly similar for elastic restraint of both the larger and smaller flanges, but the degree of distortion is slightly greater for an unrestrained smaller flange, which is consistent with the lower buckling loads for these columns.

CONCLUSIONS

A rational model for the elastic lateral-distortional buckling of a column restrained completely against translation but elastically against twist at one of its flanges has been presented. An energy-based method was invoked that reduced the buckling equation to a third order eigenproblem. In addition, the same energy-based rationale was used to develop an equation for the critical load of an elastically restrained flange in the so-called U-frame method.

The numerical method was used to investigate the buckling of doubly-symmetric and monosymmetric I-section columns with elastic twist restraint. The buckling mode with twist restraint is lateral-distortional, with the free flange, restrained only by the stiffness of the web, displacing and twisting and accompanied by distortion of the web in the plane of its cross-section. For a monosymmetric column, the buckling loads are greater when the smaller flange is the flange that is elastically restrained.

The U-frame model was compared with the numerical solutions for the doubly-symmetric and monosymmetric columns. In the former case, the predictions of the buckling load using this approximate method are reasonably accurate for the higher twist restraints, but the U-frame model is quite unconservative for monosymmetric columns.

REFERENCES


NOTATION

\( A_F \) \hspace{1cm} Area of flange
\( A_W \) \hspace{1cm} Area of web
\( B_T, B_B \) \hspace{1cm} Widths of top and bottom flanges respectively
\( [D_F] \) \hspace{1cm} Flange elastic property matrix
\( [D_W] \) \hspace{1cm} Web elastic property matrix
\( E \) \hspace{1cm} Young's modulus of elasticity
\( EI_f \) \hspace{1cm} Flexural rigidity of flange
\( EI_y \) \hspace{1cm} Minor axis flexural rigidity
\( GJ \) \hspace{1cm} Saint Venant torsional rigidity
\( [g] \) \hspace{1cm} Member stability matrix
\( [g_F], [g_W] \) \hspace{1cm} Flange and web stability matrices respectively
\( h \) \hspace{1cm} Distance between flange centroids
\( I_w \) \hspace{1cm} Second moment of area of web per unit length
\( k_t \) \hspace{1cm} Translational restraint stiffness of flange in U-frame model
\( k_z \) \hspace{1cm} Elastic twist restraint per unit length
\( [k] \) \hspace{1cm} Member stiffness matrix
\( [k_F], [k_W] \) \hspace{1cm} Flange and web stiffness matrices, respectively
\( [k_R] \) \hspace{1cm} Flange restraint stiffness matrix
\( L \) \hspace{1cm} Length of column
\( N \) \hspace{1cm} Axial load
\( N_{cr} \) \hspace{1cm} Critical load
\( N_E \) \hspace{1cm} Euler buckling load = \( \pi^2 EI_y / L^2 \)
\( n \) \hspace{1cm} Harmonic number
\( \{q\} \) \hspace{1cm} Vector of buckling degrees of freedom
\( t \) \hspace{1cm} Web thickness
$U$  Total strain energy stored during buckling
$U_F, U_W$  Flange and web contributions to $U$
$U_R$  Restraint contribution to $U$
$u_B$  Displacement of unrestrained (bottom) flange
$u_w$  Displacement of web
$V$  Total work done during buckling
$V_F, V_W$  Flange and web contributions to $V$
$v_T, v_B$  Vertical displacement of top and bottom flanges
$x, y, z$  Cartesian axis system through web mid-height
$\alpha_z$  Dimensionless twist restraint parameter
$\Delta$  Web flexibility per unit length in U-frame model
$\Delta_1$  Contribution of web flexibility to $\Delta$
$\Delta_2$  Contribution of flange twist to $\Delta$
$\{\varepsilon_F\}$  Flange generalised strain vector
$\{\varepsilon_W\}$  Web generalised strain vector
$\eta$  $y/h$
$\lambda$  Buckling load factor
$\nu$  Poisson's ratio
$\xi$  $z/L$
$\sigma$  Applied uniform stress
$\phi_T, \phi_B$  Twist of top and bottom flanges respectively.
FIG. 1  DISTORTION IN A COLUMN FULLY RESTRAINED ON ONE FLANGE
FIG. 2 TWIST RESTRAINTS TO COLUMN FLANGE
FIG. 3  BUCKLING DEFORMATIONS
(a) U - frame

(b) Elastically restrained flange

FIG. 4  "U - FRAME" MODEL
FIG. 5  BUCKLING OF DOUBLY-SYMMETRIC COLUMN
WITH h/t = 100
Fig. 6  Buckling of Doubly-Symmetric Column
with $h/t=200$
FIG. 7  BUCKLING MODES FOR DOUBLY-SYMMETRIC RESTRAINED COLUMNS
$B_1/h = \frac{1}{6}$
$B_2/h = \frac{1}{3}$
$T_1/t = T_2/t = 2$
$h/t = 100$

FIG. 8  BUCKLING OF MONOSYMMETRIC COLUMN WITH LARGER FLANGE UNRESTRANDED
FIG. 9  BUCKLING OF MONOSYMMETRIC COLUMN WITH SMALLER FLANGE UNRESTRAINED
FIG. 10  BUCKLING MODES FOR MONOSYMMETRIC RESTRAINED COLUMNS

\[ a_z = 0 \]  \[ 10 \]  \[ 100 \]  \[ 1000 \]