FREQUENCY ANALYSIS
OF TAPERED RECTANGULAR PLATES
BY THE FINITE STRIP METHOD

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SUMMARY:

In this paper, the frequency analysis of rectangular plates of variable thickness using higher order finite strips is presented. The stiffness and mass matrices of a higher order strip whose thickness variation may be defined by one coordinate polynomial of any degree in its transverse direction has been derived in a simple manner, and with the same ease as that of the conventional finite strip. Results from numerical examples are in excellent agreement with other solutions.

INTRODUCTION

The vibration of elastic plates of constant thickness has received considerable attention from various authors whose works are reported elsewhere (7). On the other hand, relatively little research has been done on the vibration of plates of variable thickness. It appears that the first attempt was reported in 1960 by Mazurkiewicz (8, 9). Since then a number of papers have been published in elsewhere (1, 2, 5, 6, 10).

In this paper, the frequency analysis of linear elastic rectangular plates of varying thickness in one direction is investigated using the finite strip method. The thickness variation of the finite strip is limited to a one-way polynomial of any degree in its transverse direction. A plate of variable thickness is therefore approximated by a series of

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strips oriented in a manner such that their longitudinal edges are perpendicular to the direction in which the plate thickness varies. The stiffness and mass matrices of the finite strip with any end conditions are derived in a very simple manner through the use of an algorithm for the exact integration of congruently transformed matrices. Such algorithm eliminates the tedious task of deriving the relevant properties of the finite strip explicitly. Also, it allows the derivation of the properties of higher order finite strips with the same ease as in the case of the basic finite strip with 4 degrees of freedom (DOF). The attractive feature of the finite strip technique is the reduction of the two-dimensional problem to a one-dimensional one due to its semi-analytical nature. This reduction will lead to the use of much smaller matrices, thereby considerably reducing the computational effort.

The finite strip has been successfully used in the vibration of plates of constant thickness (3, 4). In Reference 4, a solution for a simply supported plate with linearly varying thickness in one direction was obtained for three natural modes using two types of the conventional 4 DOF finite strip, namely: (i) a strip of constant thickness, thus approximating the plate with strips of different individual thicknesses; and (ii) strip with thickness varying linearly in its longitudinal direction and constant in its transverse direction.

Two numerical examples are presented herein to demonstrate the accuracy and versatility of the finite strip technique. The numerical results, obtained using a conventional finite strip with 4 DOF and a higher order strip with 6 DOF, are compared with other available solutions.
THEORETICAL CONSIDERATIONS

(a) Eigenvalue Equation

The free vibration equations for a plate described by a discrete coordinate system may be written as

\[
([K] - \omega^2 [M]) \{v\} = 0
\]  

(1)

in which \([K]\) represents the stiffness of the assembled structure, \([M]\) is the mass matrix of the system, \(\omega\) is the circular frequency of vibration and \(\{v\}\) is the system coordinate displacement vector.

For computation purposes, Eq.1 may be reduced to,

\[
[K^{-1}][M] \{v\} = \lambda \{v\}
\]  

(2)

where \(\lambda = 1/\omega^2\).

(b) Strain Energy and Stiffness Matrix

The assumed displacement function, \(w\), for the \(m^{th}\) harmonic of a typical strip as shown in Fig.1 is taken in the form

\[
w = [A(x)] \{a_m\} Y_m(y)
\]  

(3)

or may be rewritten in the form

\[
w = [A(x)][C^{-1}] \{q_m\} Y_m(y)
\]  

(4)

where

- \([A(x)]\) = matrix of polynomial terms in the \(x\)-direction
- \(Y_m(y)\) = the \(m^{th}\) term of a basic function series in the \(y\)-direction which satisfies a priori the two end conditions of the strip.
\{a_m\} = \text{vector of undetermined coefficients}

\{q_m\} = \text{the strip nodal displacement vector}

\[ [C] = \text{referred to as connectivity matrix defined from the relation } \{q_m\} = [C]\{a\} \]

A partial list of basic functions \(Y_m\) for different end conditions is given in Appendix I for later use. For a conventional finite strip with 4 DOF of Fig. 1(b), matrix \([A]\) and vector \(\{q\}\) in Eq. 4 are taken as,

\[
[A(x)] = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix}
\]

\[
\{q_m\} = \{w_{im}, (w'_{x})_{im}, w_{jm}, (w'_{x})_{jm}\}
\]

and for a higher order strips with 6 DOF of Fig. 1(c),

\[
[A(x)] = \begin{bmatrix} 1 & x & x^2 & x^3 & x^4 & x^5 \end{bmatrix}
\]

\[
\{q_m\} = \{w_{im}, (w'_{x})_{im}, (w'_{xx})_{im}, w_{jm}, (w'_{x})_{jm}, (w'_{xx})_{jm}\}
\]

where comma denotes differentiation with respect to the variable in the subscript.

The strain energy of linearly elastic finite strip is given by,

\[
U = \frac{1}{2} \int_A \sigma^T \{\varepsilon\} \, dA = \frac{1}{2} \int_A \{\varepsilon\}^T [D] \{\varepsilon\} \, dA
\]

where

\[
\{\sigma\} = \begin{bmatrix} M_x & M_y & M_{xy} \end{bmatrix}
\]

\[
\{\varepsilon\} = \begin{bmatrix} w_{xx} & w_{yy} & 2w_{xy} \end{bmatrix}
\]

\[
[D] = \varepsilon^3 [\bar{D}]
\]
In Eqs. 8(c) and 8(d), \( t = t(x) \) is the strip thickness, \( E \) = the modulus of elasticity and \( \mu \) = the Poisson's ratio.

Appropriate differentiation of Eq.4 as indicated in Eq.8(b), yields the generalized strain vector \( \{\varepsilon\} \),

\[
\{\varepsilon\} = [F_m][B(x)][C^{-1}]{q_m}
\]  

(9)

where

\[
[F_m] = \begin{bmatrix}
Y_m & 0 & 0 \\
0 & Y''_m & 0 \\
0 & 0 & Y'_m
\end{bmatrix}
\]  

(10)

\([B(x)] = \) a matrix of polynomial terms.

In Eq.10, the prime denotes differentiation with respect to the \( y \)-coordinate. For the 4 DOF conventional strip, matrix \([B]\) in Eq.9 is given explicitly as,

\[
[B] = \begin{bmatrix}
0 & 0 & 2 & 6x \\
1 & x & x^2 & x^3 \\
0 & 2 & 4x & 6x^2
\end{bmatrix}
\]  

(11)

and for the 6 DOF higher order strip,

\[
[B] = \begin{bmatrix}
0 & 0 & 2 & 6x & 12x^2 & 20x^3 \\
1 & x & x^2 & x^3 & x^4 & x^5 \\
0 & 2 & 4x & 6x^2 & 8x^3 & 10x^4
\end{bmatrix}
\]  

(12)
It can be shown through Eqs. 7 and 9 that the stiffness matrix, \( k \), of the strip element is given by

\[
[k_{mn}] = [C^{-1}]^T [K_{mn}] [C^{-1}]
\]  

(13)

where

\[
[K_{mn}] = \int_o^b [B]^T [D_{mn}^*] [B] \, t \, dx
\]  

(14)

\[
[D_{mn}^*] = \int_o^a [F_m]^T [ \delta ] [F_n] \, dy
\]  

(15)

\[a, b = \text{the longitudinal and transverse dimensions of the strip element, respectively.}\]

Substitution of Eq.10 into Eq.15 and evaluating the integral gives

\[
[D_{mn}^*] = \frac{E}{12(1-\mu^2)} \begin{bmatrix}
I_1 & \mu I_4 & 0 \\
I_2 & 0 & I_3 \\
\text{sym.} & \frac{1-\mu}{2} I_3
\end{bmatrix}
\]  

(16)

where

\[
I_1(m, n) = \int_o^a Y_m Y_n \, dy
\]

\[
I_2(m, n) = \int_o^a Y'''_m Y''_n \, dy
\]

\[
I_3(m, n) = \int_o^a Y'_m Y'_n \, dy
\]

\[
I_4(m, n) = \int_o^a Y''_m Y'''_n \, dy
\]  

(17)
Consider now a strip element whose thickness variation is given by

\[ t = t_i \left[ 1 + r \left( \frac{\bar{X}}{b} \right) \right] \tag{18} \]

where

\[ r = \frac{t_f - t_i}{t_i} = \text{taper ratio} \]

\[ t_i, t_f = \text{thickness of strip at nodal lines i and j, respectively.} \]

Substitution of Eq.18 into Eq.14 yields,

\[
[k_{mn}] = t_i^3 \left[ \beta_1 \int_0^b B^T D_{mn}^* B \, dx + \beta_2 \int_0^b B^T D_{nn}^* B \, dx \right. \\
+ \left. \beta_3 \int_0^b B^T D_{mn}^* B \, x^2 \, dx + \beta_4 \int_0^b B^T D_{nn}^* B \, x^3 \, dx \right] \tag{19} \]

where \( \beta_1 = 1, \beta_2 = (3r^2)/b, \beta_3 = (3r)/(b^2) \) and \( \beta_4 = (r/b)^3 \).

For other variation of the strip thickness such as parabolic, the expression for \([k_{mn}]\) would simply contain more integral terms. The algorithm for the computer evaluation of Eq.19 is outlined in Appendix II.

It must be observed that in deriving the stiffness matrices of strips of different degrees of freedom (e.g. 4 DOF, 6 DOF), only matrices \([A], [C]\) and \([B]\) need to be defined for each case.

(c) **Mass Matrix**

The inertia or mass matrix, \([m]\), of the strip can be evaluated from the expression,

\[
[m]_{mn} = \left[C^{-1}\right]^T \tilde{[m]}_{mn} \left[C^{-1}\right] \tag{20} \]
where

\[
\begin{align*}
\begin{bmatrix} \tilde{m} \end{bmatrix}_{mn} &= \phi_{mn} \int_0^b [A]^T [A] t \, dx \\
&\quad + (r/b) \int_0^b [A]^T [I] [A] x \, dx
\end{align*}
\]

(21)

where

\[
\phi_{mn} = \rho \int_0^a Y_m Y_n \, dy
\]

(22)

\[\rho = \text{mass density of the plate}\]

For a strip whose thickness varies according to Eq.18, the expression for \([\tilde{m}]\) becomes,

\[
\begin{align*}
\begin{bmatrix} \tilde{m} \end{bmatrix}_{mn} &= \phi_{mn} t_i \left[ \int_0^b [A]^T [I] [A] \, dx \right. \\
&\quad + \left. (r/b) \int_0^b [A]^T [I] [A] x \, dx \right]
\end{align*}
\]

(23)

where \([I] = (1 \times 1)\) identity matrix. With the introduction of the identity matrix in Eq.23, the same subroutine developed to evaluate Eq.19 can also be used to evaluate Eq.23. In evaluating the mass matrices of strips of different degrees of freedom, only matrices \([C]\) and \([A]\) need appropriate modifications.

**NUMERICAL EXAMPLES**

Two numerical examples are presented herein to illustrate the application of the preceding theory. Although in both examples the plates are linearly tapering in one direction, plates of other thickness variation such as quadratic or any degree polynomial in one direction, can be solved readily using finite strips whose thickness will match that of the plate.
A linearly tapered rectangular plate simply supported on all edges as shown in Fig. 2 is approximated by a series of strips which are oriented in such a way that the thickness of each strip varies linearly in its transverse direction, and constant in its longitudinal direction. In view of the simply supported end conditions of the strips, the expression for the basic function $Y_m$ is given by Eq. 24 of Appendix I. The natural frequencies of the plate were obtained using both the conventional finite strip (4 DOF) and the higher order strip (6 DOF) for two values of the plate aspect ratio, $b/a = 1, 0.5$, and the plate taper ratio, $r = 1, 0.6$. The numerical results, including those obtained by Appl and Byers (1) and by Chopra and Durvasula (5), are tabulated in Table 1.

It can be seen from Table 1 that for the plate with aspect ratio, $b/a = 1.0$, the results obtained using four higher order strips, to approximate the plate, are very close to those obtained using eight conventional strips. However, observed that the computed frequency parameters using higher order strips are slightly less than the upper bound values given by Chopra and Durvasula (5) while those obtained using the conventional strips are generally higher. Hence, the performance of higher order strips is considered better than the conventional ones even though the number of strips utilized to approximate the plate using the former is one half that using the latter. For the plates with aspect ratios, $b/a = 1$ and $0.5$, and taper ratio, $r = 0.6$, the finite strip solutions for the fundamental frequency are within the lower and upper bound values by Appl and Byers (1). In general, the finite strip solutions are in excellent agreement with
the other available solutions.

The nodal patterns of the different modes, although not pre­
sented in this paper, are found to agree with those plotted by Chopra
and Durvasula (5). The nodal lines, perpendicular to the direction in
which the plate thickness varies, gradually shift towards the edge of
lower thickness as the plate taper ratio increases.

(b) Example 2 - Linearly Tapered Cantilever Square Plate

A cantilever square plate of linearly varying thickness as
shown in Fig. 3 is analyzed using both the 4 DOF and the 6 DOF finite
strips with free-free end conditions. In this case, the basic function
series is given by Eq. 25 (Appendix I). For comparison, the numerical
results for this example and the experimental and finite element solu­
tions of Dawe (6) are presented in Table 2. Dawe's finite element
solution was obtained using a (5 x 5) grid of non-compatible 12 DOF
rectangular finite elements with linearly varying thickness. Consid­
ering the experimental results as reference values, the percentage
difference between the computed frequencies using higher order strips
and the corresponding experimental values ranges from 0 to 4½ percent.
The percentage differences quoted can be attributed to the errors in
obtaining both solutions. The finite element solution, although it
differs by no more than 3½ percent from the experimental ones, involved
matrices in the order of (90 x 90) while the finite strip solutions have
matrices of order no higher than (16 x 16).

CONCLUSIONS

The superior performance of higher order finite strips for
plate vibration has been demonstrated herein. The stiffness and mass
matrices of higher order strip whose thickness variation is defined by any
one-coordinate polynomial in its transverse direction can be derived in a simple manner, and with the same ease as that of the conventional finite strip. Thus, the tedious task of deriving the stiffness and mass matrices explicitly is eliminated.

The application of the finite strip technique to vibration analysis of plates, has the main advantage of using relatively much smaller matrices to obtain results with accuracy similar to those predicted by the more versatile finite element method. Such advantage would result in substantial saving in computation costs.
REFERENCES


APPENDIX I - BASIC FUNCTIONS

1. Both ends simply supported.

\[ Y_m = \sin \left( \mu_m y/a \right) \]
\[ \mu_m = \pi, 2\pi, \ldots, m\pi \] \hspace{1cm} (24)

2. Both ends free.

\[ Y_1 = 1, \ \mu_1 = 0 \]
\[ Y_2 = 1 - \frac{2y}{a}, \ \mu_2 = 1 \]
\[ Y_m = \left[ \sin \left( \mu_m y/a \right) + \sinh \left( \mu_m y/a \right) \right] - \alpha_m \left[ \cos \left( \mu_m y/a \right) + \cosh \left( \mu_m y/a \right) \right] \] \hspace{1cm} (25)
\[ \mu_m = 4.7300, 7.8532, 10.9960, 14.1370 \ldots \]
\[ \ldots \frac{(2m - 3)\pi}{2} (m = 3, 4, \ldots \infty) \]
\[ \alpha_m = \frac{\sin \mu_m - \sinh \mu_m}{\cos \mu_m - \cosh \mu_m} \]

3. Both ends clamped.

\[ Y_m = \left[ \sin \left( \mu_m y/a \right) - \sinh \left( \mu_m y/a \right) \right] - \alpha_m \left[ \cos \left( \mu_m y/a \right) - \cosh \left( \mu_m y/a \right) \right] \]
\[ \mu_m = 4.7300, 7.8532, 10.9960, 14.1370 \ldots \]
\[ \ldots \frac{(2m + 1)\pi}{2} \] \hspace{1cm} (26)
\[ \alpha_m = \frac{\sin \mu_m - \sinh \mu_m}{\cos \mu_m - \cosh \mu_m} \]
4. One end simply supported and the other end clamped.

\[ Y_m = \sin \left( \mu_m y/a \right) - \alpha_m \sinh \left( \mu_m y/a \right) \]

\[ \mu_m = 3.9266, 7.0685, 10.2102, 13.3520, \ldots \]

\[ \ldots \frac{(4m+1)\pi}{4} \]

\[ \alpha_m = \sin \mu_m / \sinh \mu_m \]
APPENDIX II - EVALUATION OF $[\bar{k}_{mn}]$

A typical integral in the expression of $[\bar{k}_{mn}]$ as given by Eq.19 takes the form,

$$[r_{mn}] = \int_0^b B^T D^*_m \mathbf{B} x^\alpha \, dx$$  \hspace{1cm} (28)

In matrix $\mathbf{B}$, a typical term consists of a constant coefficient, $c$, and a power of $x$-coordinate, $h$. Thus, the $k_i$-th and $l_j$-th terms of $[\mathbf{B}]$ may be written as follows:

$$\text{(ki-th term of } \mathbf{B}) = c_{ki} x^{h_{ki}}$$

$$\text{(lj-th term of } \mathbf{B}) = c_{lj} x^{h_{lj}}$$ \hspace{1cm} (29)

The constants $c_{ki}$ and the powers $h_{ki}$ are summarized in the following matrices for use as DATA statement in the computer program:

(a) For the conventional finite strip (4 DOF),

$$[c] = \begin{bmatrix} 0 & 0 & 2 & 6 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 \end{bmatrix}$$

$$[h] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$ \hspace{1cm} (30)
(b) For the higher order finite strip (6 DOF),

\[
[c] = \begin{bmatrix}
0 & 0 & 2 & 6 & 12 & 20 \\
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 2 & 4 & 6 & 8 & 10
\end{bmatrix}
\]

\[
[h] = \begin{bmatrix}
0 & 0 & 0 & 1 & 2 & 3 \\
0 & 1 & 2 & 3 & 4 & 5 \\
0 & 0 & 1 & 2 & 3 & 4
\end{bmatrix}
\]

(31)

It can be shown that the ij-th term of matrix \([r_{mn}]\)
defined by Eq.28, is given by

\[
(r_{mn})_{ij} = \sum_k \sum \kappa_i (d^*_mn\kappa_j c_j)_i . I(s)
\]

(32)

where

\[
I(s) = \int_{0}^{b} x^{s-1} dx = \frac{b^s}{s}
\]

(33)

\[
s = h_{ki} + h_{kj} + \alpha + 1
\]

For a linearly tapered finite strip, matrix \([\kappa_{mn}]\)
contains four integrals as given by Eq.19. Hence, the ij-th term
of \([\kappa_{mn}]\) is as follows:

\[
(\kappa_{mn})_{ij} = 4 \sum_{p=1}^{4} (x^p_{mn})_{ij}
\]

(34)

where p denotes the number of integrals in Eq.19. In evaluating
Eq.34, the values of \(\alpha\) for the 1st, 2nd, 3rd and 4th integrals in
Eq.19 are 0, 1, 2 and 3, respectively.
TABLE 1 - VALUES OF $\omega_{mn}^2 \sqrt{\rho h_o/D_o}$ FOR LINEARLY TAPERED RECTANGULAR PLATES SIMPLY SUPPORTED ON ALL EDGES; $\mu = 0.3$

| Plate taper ratio, $r$ | $m$ | $n$ | Chopra & Durvasula (5) | Finite Strip Solutions
<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
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</thead>
<tbody>
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<td>Conventional 4 d.o.f.</td>
</tr>
<tr>
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<td>4 strips</td>
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<td>Plate aspect ratio, $\bar{b}/\bar{a} = 1.0$</td>
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*{ } and ( ) = the upper and lower bound values for the fundamental frequency by Appl and Byers (1).
TABLE 2 - VALUES OF $\omega_{mn} a^2 \sqrt{\rho h_0 / D_0}$ FOR LINEARLY TAPERED CANTILEVER SQUARE PLATE; $\mu = 0.3$

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<th>$n$</th>
<th>Dawe (6)</th>
<th>Experimental</th>
<th>F.E*</th>
<th>Finite Strip Solutions</th>
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<th>H.O. 6 d.o.f</th>
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<td>4 strips</td>
<td>8 strips</td>
<td>4 strips</td>
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<td>1</td>
<td>4.069</td>
<td>4.083</td>
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<td>4.158</td>
<td>4.158</td>
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<tr>
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<td>3</td>
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<tr>
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<td>4</td>
<td>-</td>
<td>-</td>
<td>76.945</td>
<td>73.149</td>
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*Finite element solution using 25 linearly tapered non-compatible 12 d.o.f. rectangular elements.

H.O. = higher order
FIGURE 1: PLATE DIVIDED INTO FINITE STRIPS, CONVENTIONAL AND HIGHER ORDER FINITE STRIPS.

FIGURE 2: LINEARLY TAPERED RECTANGULAR PLATE SIMPLY SUPPORTED ON ALL EDGES (EXAMPLE 1).

FIGURE 3: LINEARLY TAPERED CANTILEVER SQUARE PLATE (EXAMPLE 2).
FIGURE 1

(a) Plate divided into finite strips

(b) Conventional finite strip (4 DOF)

(c) Higher order finite strip (6 DOF)

(d) Section 1-1

(e) Section 2-2
(a) Plate divided into strips.

FIGURE 2

(b) Section 1-1

Parameters investigated:

\( \frac{b}{a} = 1, 0.5 \)

\( r = 1, 0.6 \)

\( \mu = 0.3 \)

FIGURE 3

Parameters investigated:

\( \frac{\tilde{b}}{\tilde{a}} = 1 \)

\( r = -20/7 \)

\( \mu = 0.3 \)