IN-PLANE BUCKLING OF SHEAR
DEFORMABLE CIRCULAR
RINGS AND ARCHES

by

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Circular Arches; Buckling; Shear deformations;

A finite strain formulation is developed for elastic circular arches and rings in which the effects of shear deformations are included in addition to the bending and the axial deformations. Timoshenko beam hypothesis is adopted for incorporating shear. Finite strains are defined in terms of the normal and shear component of the longitudinal stretch. The constitutive relations for stress and finite strain are based on a hyperelastic constitutive model. Virtual work and equilibrium equations are derived. Closed form buckling solutions are derived for rings under hydrostatic pressure and circular high arches under radially directed dead pressure. In both cases the effects of axial deformation prior to buckling and shear deformations are included. The formulation developed herein is validated through comparisons with special cases and solved examples from the literature. Some practical cases are selected to show the importance of including the effects of shear deformations for deep arches.
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1. Introduction

A critical failure mode for elastic arches and rings is in-plane buckling. Classical linearized in-plane buckling solutions for ring and circular arches and a thorough literature review on the early studies of arches can be found in the books of Timoshenko and Gere (1963), Vlasov (1959), Simitse (1976), Simitse and Hodges (2006) and Bazant and Cedolin (2003). Several analytical solutions that consider second order effects due to geometric nonlinearities have also been developed which includes the works of Schreyer and Masur (1966), DaDeppo and Schmidt (1969), DaDeppo and Schmidt (1972), DaDeppo and Schmidt (1975), Oran and Reagan (1969), Oran and Bayazit (1978), Pi and Trahair (1998) and Hodges (1999). Batoz (1979) presented a short communication investigating the importance and use of the correct nonlinear strain-displacement relations for the problem of a simple hinged circular arch under constant “dead” pressure. Pi, Bradford et al. (2002) and Bradford, Uy et al. (2002) obtained closed form solutions for their second order nonlinear formulation and showed the importance of including the effects of prebuckling deformations in order to predict the buckling loads correctly. Analytical solutions for shallow and non-circular arches were also presented in Chen and Lin (2005), Moon, Yoon et al. (2007). Gengshu, Pi et al. (2008) included the effects of transverse stresses and also discussed various buckling formulas derived for the arch buckling problem of a uniform radial load. Discrepancies between several formulations occur in the literature and these discrepancies are attributed to the differences to the accuracy of the definition of finite strain in Chang, Kim et al. (1996), Gengshu, Pi et al. (2008). Formulations that have considered the shear deformation effects in arch buckling analysis include Reissner (1972), Goto, Yoshimitsu et al. (1990), Chang, Kim et al. (1996) and Gengshu, Pi et al. (2008). A geometrically exact finite strain formulation that considers the stretch bending coupling effects in circular high arches is developed by Hodges (1999). However in Hodges’ formulation shear deformations are not included.

Effects of shear deformation in buckling and geometrically nonlinear analysis have been a subject of debate. There have been discussions on whether the Harinx or Engesser formulation is correct for the elastic buckling of shear deformable columns. In the current paper a stretch based approach is used to develop the curvature and finite strains for circular arches which are consistent with the formulation in Hodges (1999) when the shear deformation effects are excluded. The developed formulation is validated through comparisons with solved examples from the literature for which closed form solutions are obtained. The effects of shear deformations for the analysis of deep arches are also illustrated.

An outline of the contents of this paper is as follows. In Section 2, the kinematics for a circular arch are developed using the Timoshenko beam analogy. The cross-sectional displacements are defined using displacements of a reference axis and a bending finite rotation. The nonlinear normal and shear finite strain expressions are derived based on the relative stretch taken normal to the displacement cross-sectional plane and taken tangential to
the cross-section plane, respectively. Hyperelastic constitutive relations for stresses and internal actions are derived using the strain energy density for a compressible isotropic neo-Hookean material proposed in Attard and Hunt (2004). The constitutive relationships are written in terms of Lagrangian physical stresses. Section 3 details the virtual work and equilibrium equations for in-plane behavior of circular arches under applied conservative tractions. In Section 4, the prebuckling linearized solution for the displacements and internal actions are derived for a prismatic circular arch under a radially distributed conservative load. Section 5 looks at the buckling of a ring under hydrostatic pressure including the effects of shear while Section 6 presents the closed form solution for the buckling of a circular arch under radially directed conservative distributed load. Expressions for the second variation of the total potential are developed in Section 7. Finally, Section 8 provides a summary of the work in this paper. Further detailed derivations needed to support the work in this paper are presented in the Appendices A and B.

2. Displacement Model for a Circular Arch, including Bending, Shear and Axial Deformation

Consider a circular arch with a prismatic cross-section having a reference axis of radius $R_o$ as shown in Figure 1. The position vector $s$ of a point on the material lines of the arch before deformation can be described using the polar coordinates $r$ and $\phi$ (refer to Figure 1), thus

\[ s = (R_o + r)[\cos \phi \mathbf{e}_1 + \sin \phi \mathbf{e}_2] \]  \hspace{1cm} (1)

with $r$ measured from the reference axis. In general, a scalar quantity will be represented by a lowercase italic light symbol while a bold lower case symbol such as $\mathbf{u}$ will be used to represent a vector. The symbols $\mathbf{e}_1$ and $\mathbf{e}_2$ denote unit base vectors associated with each of the two axes of a planar rectangular reference frame as shown in Figure 1. Unlike a straight beam each of the material lines along the beam’s length are of different lengths. The covariant base and contravariant base vectors calculated using Eqn (1) are:

\[ g_r = \frac{\partial s}{\partial r} = \cos \phi \mathbf{e}_1 + \sin \phi \mathbf{e}_2 \quad g^r = \cos \phi \mathbf{e}_1 + \sin \phi \mathbf{e}_2 \]  \hspace{1cm} (2)

\[ g_\phi = \frac{\partial s}{\partial \phi} = (R_o + r)[-\sin \phi \mathbf{e}_1 + \cos \phi \mathbf{e}_2] \quad g^\phi = \frac{-\sin \phi \mathbf{e}_1 + \cos \phi \mathbf{e}_2}{(R_o + r)} \]  \hspace{1cm} (3)

The metric tensor components for the undeformed shape are therefore

\[ g_{rr} = 1 \quad g_{\phi \phi} = (R_o + r)^2 \quad g_{r\phi} = 0 \quad g^{rr} = 1 \quad g^{\phi \phi} = \frac{1}{(R_o + r)^2} \quad g^{r\phi} = 0 \]  \hspace{1cm} (4)

Both the covariant and contravariant set of base vectors are orthogonal since $g_{r\phi} = 0$ and $g^{r\phi} = 0$. The Timoshenko Beam hypothesis for beam bending with shear is adopted here. Any planar cross-section perpendicular to the initial reference axis will remain planar but is not perpendicular to the reference axis as the beam deforms under load. The cross-sections
perpendicular to the reference axis are assumed to undergo no strain within the cross-section plane. The rotation angle between the tangent base vector \( \hat{g}_\phi \) in the deformed state and the initial tangent base vector \( g_\phi \) in the undeformed state is denoted by \( \theta + \phi_0 \), which consists of a bending component \( \theta \) and a shear component \( \phi_0 \).

![Figure 1 Circular Arch](image)

Two displacements are required to describe the deformation. The covariant components are denoted by \( u_r \) and \( u_\phi \). The displacement component \( u_r \) is in the direction of \( r \) while the displacement component \( u_\phi \) is tangential to the reference axis fibers of the arch. The \( u_r \) component is given by

\[
u_r = u_r = v(\phi) - r(1 - \cos \theta(\phi))\]

In the above, \( v(\phi) \) is the displacement of the reference axis in the \( r \) direction. The angle \( \theta(\phi) \) is the bending rotation of the cross-section. The covariant and contravariant displacement components \( u_r \) and \( u_\phi \) are equal because the covariant and contravariant base vectors are the same. Since \( u_r \) and \( u_\phi \) are equal and \( g_\phi = (R_0 + r)^2 g_\phi \), it follows that

\[
u_\phi g_\phi = u_\phi g_\phi \quad u_\phi = (R_0 + r)^2 u_\phi \]

To determine the strain and relative elongation we will need the covariant derivatives of the displacement components and hence will need the Christoffel symbols. For the polar coordinate system considered, the Christoffel symbols are:

\[
\Gamma^r_{rr} = \Gamma^r_{r\phi} = \Gamma^r_{\phi r} = 0 \quad \Gamma^\phi_{rr} = -\frac{1}{(R_0 + r)} \quad \Gamma^r_{\phi\phi} = \Gamma^\phi_{\phi r} = \frac{1}{(R_0 + r)}
\]

The covariant derivatives are then:
\[
\begin{align*}
&\left. u_r \right|_r = u'_r = u'_{r,r} = u_{r,r} \\
&\left. u_r \right|_r = u_{r,r} - \frac{u_\theta}{(R_o + r)} \\
&\left. u'_r \right|_b = u_{r,b} - \frac{u_\theta}{(R_o + r)} \\
&\left. u_\phi \right|_r = u_\phi - \frac{u_\theta}{(R_o + r)} \\
&\left. u'_{\phi} \right|_r = u_{\phi,r} - \frac{u_\theta}{(R_o + r)} \\
&\left. u''_{\phi} \right|_r = u_{\phi,r} + \frac{u_\theta}{(R_o + r)} \\
&\left. u_{\phi,\phi} \right|_r = u_{\phi,\phi} + \frac{u_\theta}{(R_o + r)} \\
&\left. u_{\phi,\phi} \right|_r = u_{\phi,\phi} + \frac{u_\theta}{(R_o + r)} \\
\end{align*}
\]

(8)

In the above, \( u_{r,r} \) symbolises differentiation of \( u_r \) with respect to \( r \). Assuming that there is no strain within the plane of the cross-sections then \( \lambda_r \) (see Figure 2) is constrained to unity, therefore we must have

\[
\lambda_r = \sqrt{1 + \frac{2\left. u_r \right|_r + \left. u'_r \right|_r + \left. u''_r \right|_r + \left. u_{\phi,\phi} \right|_r}{g_{rr}}} = 1
\]

(9)

Substituting Eqns (5) and (8) into Eqn (9) and solving for \( u_\phi \) gives:

\[
\frac{u_\phi}{(R_o + r)} = (R_o + r)u_\phi = u_\phi(\phi) - r \sin(\theta) \phi
\]

(10)

where \( u_\phi(\phi) \) is the tangential displacement of the reference axis. The displacement vector is therefore:

\[
u = \left[ \frac{u_r - r \sin(\Phi)}{(R_o + r)} \right]g_r + \left[ v - r(1 - \cos(\phi)) \right]g_r
\]

\[
u = \left[ \cos(\phi - \theta) - \cos(\phi) \right]e_1 + r\left[ \sin(\phi - \theta) - \sin(\phi) \right]e_2
\]

(11)

In the above, \( u_c \) is the displacement of the reference axis given by:

\[
u = \left[ \frac{-u_\phi \sin(\phi) + v \cos(\phi)}{(R_o + r)} \right]g_r + \left[ u_\phi \cos(\phi) + v \sin(\phi) \right]e_2
\]

(12)

The covariant tangent base vectors in the deformed state can now be determined using:

\[
\hat{g}_i = \left( \delta_i^j + u'_j \right)g_j
\]

(13)

Substituting Eqns (5) and (10) into Eqn (13), the covariant tangent base vectors in the deformed state are then:
\[ \hat{g}_s = \left(1 + u_s^s \right) g_s + u_s^s \left| g_s \right| \]
\[ = \frac{1}{1 + \frac{r}{R_o}} \left(1 + \bar{u}_{o,s} + \frac{r}{R_o} \cos \theta \left(1 - \frac{d\theta}{d\phi} \right) \right) g_s + R_o \left( \bar{v}_s + \frac{r}{R_o} \sin \theta \left(1 - \frac{d\theta}{d\phi} \right) \right) g_r \] (14)

\[ \hat{g}_r = u^r \left| g_r \right| g_r + \left(1 + u^r \right) g_r = \left( -\frac{\sin \theta}{R_o + r} \right) g_s + \cos \theta g_r \] (15)

where

\[ \bar{u}_{o,s} = \frac{u_{o,\phi} + v}{R_o} \quad \bar{v}_s = \frac{v_{,\phi} - u_{o}}{R_o} \] (16)

The differentiation such as in \( \bar{v}_s \) is with respect to the undeformed profile coordinate of the reference axis, so that \( ds = R_o d\phi \). The unit normal to the cross-sectional plane is denoted by \( \hat{n} \) and can be determined from Eqn (15), that is:

\[ \hat{n} = \left( \frac{\cos \theta}{R_o + r} \right) g_s + \sin \theta g_r \] (17)

Using Eqns (14) & (15), the relative stretch in the tangential direction of the reference axis is therefore:

\[ \lambda_{\phi o} = \sqrt{\frac{\hat{g}_s}{g_{s, \phi o}^o}} = \sqrt{\left[1 + \bar{u}_{o,s} \right]^2 + \left[ \bar{v}_s \right]^2} \] (18)

Figure 3 shows a differential segment of the reference axis of length \( ds \) which undergoes deformation so that point A moves to A' and point B moves to B'. The relative stretch of the reference axis \( \lambda_{\phi o} \) is shown in Figure 3. Also displayed is the orientation and definition of the displacement derivatives \( \bar{u}_{o,s} \) & \( \bar{v}_s \) which are taken in the tangential and radial directions, respectively, defined at A.
From Eqns (4), (5), (8) and (10) we can write at the reference axis \((r=0)\):

\[
\begin{align*}
  r = 0 & \quad \hat{g}_\theta \cdot g_\phi = \lambda_{g_{\phi\phi}} \left( R_\theta \right)^2 \cos(\theta + \phi) = g_{\phi\phi} + u_{\phi\phi} = \left( R_\theta \right)^2 \left( 1 + \bar{u}_{\theta,\phi} \right) \\
  r = 0 & \quad \hat{g}_\theta \cdot g_r = R_\theta \lambda_{g_{\phi r}} \sin(\theta + \phi) = g_{r\phi} + u_{r\phi} = R_\theta \tilde{v}_s
\end{align*}
\]  

(19)  

(20)

The above equations lead to the following:

\[
\begin{align*}
  \lambda_{g_{\phi\phi}} \cos(\theta + \phi) &= 1 + \bar{u}_{\theta,\phi} \\
  \lambda_{g_{\phi\phi}} \sin(\theta + \phi) &= \tilde{v}_s
\end{align*}
\]  

(21)

\[
\begin{align*}
  \lambda_{g_{\phi r}} \cos \phi &= \left( 1 + \bar{u}_{\theta,\phi} \right) \cos \theta + \left( \tilde{v}_s \right) \sin \theta \\
  \lambda_{g_{\phi r}} \sin \phi &= -\left( 1 + \bar{u}_{\theta,\phi} \right) \sin \theta + \left( \tilde{v}_s \right) \cos \theta
\end{align*}
\]  

(22)

Hence the tangent at the deformed reference axis is:

\[
\tan(\theta + \phi) = \frac{\left( v_{,\phi} - u_\phi \right)}{\left( R_\theta + u_{,\phi} + v_\phi \right)} = \frac{\tilde{v}_s}{\left( 1 + \bar{u}_{\theta,\phi} \right)}
\]  

(23)

We can see from Eqn (23) that when there is no bending and shear rotation then

\[
\text{if} \quad \theta + \phi = 0 \implies \tilde{v}_s = 0 \implies u_\phi = v_\phi
\]  

(24)

Equation (23) can also be used to establish the curvature \( \frac{d\theta}{ds} \) that is:
\[ \frac{d\theta}{ds} + \frac{d\phi_o}{ds} = \left( \frac{\left[ v_{,\theta} - u_{,\theta} \right] R_o + v + u_{,\theta} - \left[ v_{,\phi} - u_{,\phi} \right] u_{,\phi} + v_{,\phi}}{\left( \lambda_{\phi o} \right)^2 R_o^3} \right) \]

\[ = \left( \frac{\ddot{v}_{,\theta} \left[ 1 + \ddot{u}_{,\theta} \right] - \ddot{v}_{,\phi} \ddot{u}_{,\phi}}{\left( \lambda_{\phi o} \right)^2} \right) \]

This agrees with Hodges (1999) if we exclude the shear term \( \frac{d\phi_o}{ds} \). For the case of axial inextensibility, when the normal component of the reference axis stretch is restrained to unity (see Eqn (21)) we have:

\[ \lambda_{\phi o} \cos(\theta + \phi_o) = 1 \rightarrow \ddot{u}_{,\theta} = 0 \rightarrow u_{,\phi} + v = 0 \rightarrow u_o = \int^{-\phi} \ddot{v} d\phi \]  

(26)

The curvature would then simplify to:

\[ \frac{d\theta}{ds} + \frac{d\phi_o}{ds} = \frac{v_{,\phi} + v_{,\theta}}{\left( \lambda_{\phi o} \right)^2 R_o^2} \]

(27)

Employing Eqns (14), (15) & (17), the physical normal and shear components of the longitudinal stretch (see Figure 2) are given by:

\[ \lambda_{n\phi} = \frac{\hat{n} \cdot \hat{g}_{,\phi}}{\sqrt{g_{,\phi}}} = \frac{1}{\sqrt{1 + \frac{R}{R_o}}} \left[ \lambda_{\phi o} \cos \phi_o \right] \]

(28)

\[ \lambda_{s\phi} = \frac{\hat{g}_{,r} \cdot \hat{g}_{,\phi}}{\sqrt{g_{,\phi}}} = \frac{\lambda_{\phi o} \sin \phi_o}{\sqrt{1 + \frac{R}{R_o}}} \]

(29)

Expanding Eqns (28) and (29) to second order, gives for the normal and shear strains:

\[ \lambda_{n\phi} - 1 \equiv \frac{1}{r} \left[ \ddot{u}_{,\theta} + \frac{1}{2} \ddot{v}_{,\theta}^2 - \frac{1}{2} \left( \ddot{\phi}_o \right)^2 - r \left[ \ddot{v}_{,\phi} \left( 1 - \ddot{u}_{,\phi} \right) - \ddot{v}_{,\phi} \ddot{u}_{,\phi} - \ddot{\phi}_{,\phi} - \ddot{\phi}_{,\phi} \right] \right] \]

(30)

\[ \lambda_{s\phi} \equiv \phi_o \left( 1 + \ddot{u}_{,\phi} \right) \]

(31)
2.1 Stresses

Although the deformation is linear (see Eqn (11)), the relative elongations are hyperbolic because the fibers through the depth of the arch cross-section have different lengths. The constitutive law for the physical Lagrangian stresses normal $S^{\phi\phi}$ and tangential $S^{\nu\tau}$ to the beam cross-section, respectively, are given by:

$$S^{\phi\phi} = E \left( \frac{\dot{\lambda}_{\phi\phi}}{\lambda_{\phi\phi}} - 1 \right) \quad S^{\nu\tau} = G \lambda_{\phi\phi}$$  \hspace{1cm} (32)

where $E = 2G + \Lambda$, $\Lambda = \frac{2G\eta}{(1 - 2\eta)}$ is the Lamé constant and $\eta$ is the Poisson’s ratio. These stresses are in agreement with Reissner’s proposal for beam actions (see Reissner (1972), Attard (2003)). The material parameter governing the normal stress is not the elastic modulus $E$ as would be expected for a uniaxial stress state. This is because the assumed two dimensional displacements restrain the dilation of the cross-section shape which is associated with lateral stresses not present under a uniaxial stress state (see Attard (2003), Attard and Hunt (2008)). A further approximation in beam theory is to replace $\dot{E}$ by $E$ in Eqn (32). The constitutive relationships for the internal actions can then be determined by defining the internal actions as the stress resultants over the cross-section (see Attard (2003) & Attard and Hunt (2008)) thus:

$$S^{\phi\phi} = \frac{E}{1 + \frac{r}{R_o}} \left( \frac{\dot{\lambda}_{\phi\phi} \cos \varphi_o - 1}{\lambda_{\phi\phi}} \right) - r \frac{d\varphi}{ds}$$

$$S^{\nu\tau} = \frac{G}{1 + \frac{r}{R_o}} \dot{\lambda}_{\phi\phi} \sin \varphi_o$$  \hspace{1cm} (33)
2.2 Stress Resultants

The internal actions on the cross-sectional plane are defined by:

\[ N = \int_A S^\phi dA \quad Q = \int_A S^\psi dA \quad M = \int_A -rS^\theta dA \]  \hspace{1cm} (34)

Here, \( N \) is the axial force defined perpendicular to the cross-sectional plane, \( Q \) is the shear force along the cross-sectional plane and \( M \) is the bending moment defined by the stresses normal to the cross-sectional plane. The location of the reference axis origin of the reference coordinate \( r \) is selected so that the following is satisfied.

\[ \int_A \frac{r}{1 + \frac{r}{R_o}} dA = 0 \]  \hspace{1cm} (35)

Hence, the geometric properties of area, second moment of area and radius of gyration for the cross-section are defined by:

\[ \tilde{A} = \int_A \frac{1}{1 + \frac{r}{R_o}} dA \quad \tilde{I} = \int_A \frac{r^2}{1 + \frac{r}{R_o}} dA \]  \hspace{1cm} (36)

Substituting the Eqns (33) into Eqns (34) gives for the internal actions the following:

\[ N = EA \left( \lambda_{y0} \cos \phi_o - 1 \right) \quad Q = GA \lambda_{y0} \sin \phi_o \quad M = E\tilde{I} \frac{d\theta}{ds} \]  \hspace{1cm} (37)

In the above we have replaced \( GA \) by \( GA \), where \( \tilde{A} \) is the shear area. The constitutive relationships for the internal actions to second order terms are therefore:

\[ N = E\tilde{A} \left( \tilde{\alpha}_{y} + \frac{1}{2} \tilde{v}_{ss}^2 - \frac{1}{4} (\phi_o)^2 \right) \quad Q = G\tilde{A} \phi_o (1 + \tilde{\alpha}_{y}) \]  \hspace{1cm} (38)

\[ M = E\tilde{I} \left( \tilde{v}_{ss} (1 + \tilde{\alpha}_{y}) - \tilde{v}_{yy} \right) \]  \hspace{1cm} (39)

The physical stresses can be rewritten in terms of the internal actions, hence:

\[ S^{\phi} = \frac{1}{1 + \frac{r}{R_o}} \left( \frac{N}{A} - \frac{Mr}{\tilde{I}} \right) \]  \hspace{1cm} (39)
3 Virtual Work

The virtual work $\delta W$ in terms of the physical stresses is given by:

$$\delta W = \int_V \left[ S^{\phi\phi} \delta (\lambda_{\phi\phi}) + S^{\phi r} \delta (\lambda_{\phi r}) \right] dV - \int_S [p \cdot \delta u] dS$$ (40)

with $V$ being the volume in the undeformed state, $S$ the surface where the externally applied conservative traction vector $p$ acts, kinematically admissible variations denoted by the symbol $\delta$ and displacement vector $u$. The volume differential can be expressed in terms of the differential of the profile coordinate $ds$, that is

$$dV = (R_o + r) d\phi dA = \left(1 + \frac{r}{R_o}\right) ds dA$$ (41)

Here we consider conservative uniformly distributed loads $p^{\phi}$ and $p^{r}$ in the tangential and radial directions, respectively, applied at the level of the reference axis as shown in Figure 4. Note $p^{\phi}$ and $p^{r}$ have been defined here as physical quantities for convenience. The loading vector is then given by

$$p = \frac{p^{\phi}}{R_o + r} g_{\phi} + p^{r} g_{r}$$ (42)

Substituting Eqns (34), (39) and (42) into the (40), integrating over the cross-section results in:
\[ \begin{aligned}
&\int_{0}^{L} [N \delta x_{\phi} + \lambda_{\psi} Q_{x} \delta \phi_{x} + M \delta \theta_{x} - p^{\psi} \delta u_{x} - p^{\theta} \delta v] \, ds = 0 \\
&\int_{0}^{L} [N \delta x_{\phi \phi} + Q \delta \phi_{x} + M \delta \theta_{x} - p^{\psi} \delta u_{x} - p^{\theta} \delta v] \, ds = 0 \\
&\int_{0}^{L} \left[ P_{\phi} \left( \delta u_{x} + \frac{\delta v}{R_{o}} \right) + P_{r} \left( \delta v_{x} - \frac{\delta u_{x}}{R_{o}} \right) - \lambda_{\psi} Q_{x} \delta \theta + M \delta \theta_{x} - p^{\psi} \delta u_{x} - p^{\theta} \delta v \right] \, ds = 0
\end{aligned} \] (43)

in which

\[ P_{r} = N \cos \theta - Q \sin \theta \quad P_{\phi} = N \sin \theta + Q \cos \theta \] (44)

\[ N_{i} = P_{r} \cos(\theta + \phi) + P_{\phi} \sin(\theta + \phi) = N \cos \phi_{o} + Q \sin \phi_{o} \]

\[ Q_{i} = -P_{r} \sin(\theta + \phi) + P_{\phi} \cos(\theta + \phi) = -N \sin \phi_{o} + Q \cos \phi_{o} \] (45)

\( P_{r} \) and \( P_{\phi} \) are the internal force resultants in the radial and tangential directions respectively, \( L \) is the length of the arch, while \( N_{i} \) is the axial force resultant tangential to the reference axis of the beam and \( Q_{i} \) is the shear resultant perpendicular to the reference axis. Integrating Eqn (43) by parts gives the following equilibrium equations:

\[ P_{r} + \frac{dP_{r}}{d\phi} = -p^{\psi} R_{o} \quad P_{\phi} - \frac{dP_{\phi}}{d\phi} = p^{\theta} R_{o} \quad \frac{dM}{ds} = -\lambda_{\psi} Q_{i} \] (46)

The solutions for the internal forces \( P_{r} \) and \( P_{\phi} \) are therefore

\[ P_{r} = C_{1} \sin(\phi) - C_{2} \cos(\phi) - p^{\psi} R_{o} \quad P_{\phi} = C_{1} \cos(\phi) + C_{2} \sin(\phi) + p^{\theta} R_{o} \] (47)

\[ C_{1} = (P_{r_{o}} - p^{\psi} R_{o}) \cos(\phi_{o}) + (P_{\phi_{o}} + p^{\psi} R_{o}) \sin(\phi_{o}) \]

\[ C_{2} = (P_{r_{o}} - p^{\phi} R_{o}) \sin(\phi_{o}) - (P_{\phi_{o}} + p^{\theta} R_{o}) \cos(\phi_{o}) \]

Where \( P_{r_{o}} \) & \( P_{\phi_{o}} \) are reactions evaluated at \( \phi = \phi_{o} \). The internal force actions \( N \) & \( Q \) can thus be represented in the form:

\[ N = C_{1} \cos(\phi - \theta) + C_{2} \sin(\phi - \theta) + p^{\psi} R_{o} \cos(\theta) - p^{\theta} R_{o} \sin(\theta) \]

\[ Q = C_{1} \sin(\phi - \theta) - C_{2} \cos(\phi - \theta) - p^{\psi} R_{o} \sin(\theta) - p^{\theta} R_{o} \cos(\theta) \] (48)

Incorporating Eqns (21), (44) and (45), we also have for the equilibrium equations:

\[ \text{Incorporating Eqns (21), (44) and (45), we also have for the equilibrium equations:} \]
\[ N \left( \frac{1}{R_o} - \frac{d\theta}{ds} \right) - \frac{dQ}{ds} = \left( -p^s \sin \theta + p' \cos \theta \right) \tag{49} \]

\[ Q \left( \frac{1}{R_o} - \frac{d\theta}{ds} \right) + \frac{dN}{ds} = \left( -p^s \cos \theta - p' \sin \theta \right) \tag{50} \]

\[ \frac{dM}{ds} = -\dot{\lambda}_{g0} Q_t = N \dot{\lambda}_{g0} \sin \varphi_o - Q \dot{\lambda}_{g0} \cos \varphi_o \tag{51} \]

Equations (48) satisfy Eqns (49) & (50) leaving the solution Eqn (51) to provide the complete solution to any nonlinear problem. Substituting the constitutive relationships Eqns (37) into Eqn (51), gives:

\[ E I \frac{d^2\theta}{ds^2} = N \dot{\lambda}_{g0} \sin \varphi_o - Q \dot{\lambda}_{g0} \cos \varphi_o = Q \left[ \frac{N}{GA} - \left( 1 + \frac{N}{EA} \right) \right] \tag{52} \]

Goto, Yoshimitsu et al. (1990) provided closed-form solutions with integral expressions for the elastica problem with axial and shear deformations set in a format similar to Eqn (52). Goto, Yoshimitsu et al. (1990) reduced the elliptic integrals to Legendre-Jacobi’s normal forms in order to utilize numerical computation. Closed form solutions to the in-plane arch problem will not be provided here but will be the subject of further work examining shallow arches and the effects of preliminary deformations on the buckling of high arches with shear.

Figure 5 Statically Determinate Arch under Compressive Radial Conservative Loading
4. Prebuckling Linearised Solution

Consider a prismatic circular arch as shown in Figure 5 with a subtended angle of \( \phi \) under a radially applied conservative distributed load \( p' = -p_\circ \). Here, we look at the linearised solution for this problem and examine the axial force variation with the subtended angle \( \pi - 2\phi \). The linearised constitutive relationships for the internal actions obtained from Eqns (38) are:

\[
N = EA\bar{u}_{xx} = p_\circ R_\circ \quad Q = GA\varphi_\circ \quad M = EI(\bar{v}_m - \varphi_{xx})
\]  

(53)

The linearised equilibrium equations derived from Eqns (49) to (51) are

\[
N - \frac{dQ}{d\phi} = p' R_\circ \quad Q + \frac{dN}{d\phi} = 0 \quad \frac{dM}{d\phi} = -R_\circ Q
\]

(54)

Solving the equilibrium equations gives for the internal actions:

\[
N = p' R_\circ + C_1 \sin \phi + C_2 \cos \phi \quad Q = -C_1 \cos \phi + C_2 \sin \phi \quad M = R_\circ \left[ N - p' R_\circ + C_3 \right]
\]

(55)

where \( C_1, C_2, \text{ and } C_3 \) are constants. For the case of zero bending moments at the supports, such as for a pinned or roller connection, Eqns (55) become

\[
N = p' R_\circ + C_1 \sin \phi \quad Q = -C_1 \cos \phi \quad M = C_1 R_\circ \left[ \sin \phi - \sin \phi_0 \right]
\]

(56)

All in terms of one unknown constant, \( C_1 \). Consider the case shown in Figure 4 with rollers at both supports, the shear at the supports is zero and hence \( C_1 = 0 \), therefore

\[
N = p' R_\circ = -p^\circ R_\circ \quad Q = 0 \quad M = 0
\]

(57)

The loading induces an axial compressive load in the tangential direction of \( P_\circ = -p^\circ R_\circ \). The deformations produce no bending or shear. The solutions to Eqns (53) for the deflections are then that there is a constant radial deflection, such that:

\[
v = \frac{-p_\circ R_\circ^2}{EA} \quad u_\circ = 0 \quad \varphi_\circ = 0 \quad \bar{u}_{xx} = \frac{-p_\circ R_\circ}{EA} \quad \bar{v}_m = 0
\]

(58)

Now, let’s turn our attention to the problem of pinned supports, as in Figure 11. The resulting expressions for the internal actions and the horizontal reaction force \( H \) (see Figure 11) are
\[ N = -p^e R_o \left[ 1 + \frac{4 \sin \phi \cos \phi_o}{K} \right] \]  
\[ Q = \frac{4 p^e R_o \cos \phi \cos \phi_o}{K} \]  
\[ M = \frac{-4 p^e R_o^2 \cos \phi_o}{K} \left[ \sin \phi - \sin \phi_o \right] \]  
\[ H = -p^e R_o \left[ \sin \phi_o + \frac{4 \cos \phi_o}{K} \right] \]

in which

\[ K = \left[ 3 \frac{R_o^2}{r^2} + \left( \frac{E \tilde{A}}{G A_i} - 1 \right) \right] \sin 2 \phi_o - \left( \pi - 2 \phi_o \right) \left[ \left( \frac{E \tilde{A}}{G A_i} + 1 \right) + \frac{R_o^2}{r^2} \left( 2 - \cos 2 \phi_o \right) \right] \]

If the supports are fully fixed then the internal actions are

\[ N = -p^e R_o \left[ 1 + \frac{4 \left( \pi - 2 \phi_o \right) \sin \phi \cos \phi_o}{K_{ff}} \right] \]  
\[ Q = \frac{4 p^e R_o \left( \pi - 2 \phi_o \right) \cos \phi \cos \phi_o}{K_{ff}} \]  
\[ M = \frac{-4 p^e R_o^2 \cos \phi_o}{K_{ff}} \left[ \left( \pi - 2 \phi_o \right) \sin \phi - 2 \cos \phi_o \right] \]

in which

\[ K_{ff} = \frac{8 R_o^2}{r^2} \cos^2 \phi_o - \left( \pi - 2 \phi_o \right) \sin 2 \phi_o \left( \frac{R_o}{r} \right)^2 \left( \frac{E \tilde{A}}{G A_i} - 1 \right) - \left( \pi - 2 \phi_o \right)^2 \left( \frac{R_o}{r} \right)^2 + \left( \frac{E \tilde{A}}{G A_i} + 1 \right) \]

One of the assumptions made when undertaking a buckling analysis of an arch is to assume the axial force is constant through the arch and equal to the nominal value \( p^e R_o \). Here we investigate this assumption for two values of the axial to shear rigidity ratio \( \frac{E \tilde{A}}{G A} \) taken as either 3 to represent solid sections and 25 for built-up sections with relatively low shear rigidity (see Bazant and Cedolin (2003)). Figure 6 shows the distribution of axial force through an arch with simple supports for a selected slenderness of \( \frac{R_o}{r} = 60 \). Figure 7, Figure 8, Figure 9 and Figure 10 show the variation of the axial force taken at the crest of an arch.
\( \phi = \frac{\pi}{2} \) as a function of the slenderness ratio \( \frac{R_e}{\bar{f}} \) for various subtended arch angles for an arch with either simple supports or fixed supports. The degrees given in the figures refer to the angle \( \phi_0 \) defined in Figure 5. An angle of \( \phi_0 = 0^\circ \) refers to a complete semi-circular arch. In the case of pinned supports, the axial force is approximately equal to the nominal value of \( p^0 R_e \) when \( \phi_0 \leq 45^\circ \) (the subtended angle of the arch is less than or equal to 90°) and the slenderness ratio \( \frac{R_e}{\bar{f}} \) is greater than about 60. The axial force is closer to the nominal value for the axial to shear rigidity ratio of 25. Figure 6 shows that at a slenderness ratio of \( \frac{R_e}{\bar{f}} = 60 \), the axial force distribution is almost constant through the arch span for \( \phi_0 \leq 45^\circ \) and independent of the two axial to shear rigidity ratios used in this study. For an arch with fixed supports, the compressive axial force is equal to the nominal value again when \( \phi_0 \leq 45^\circ \) but at the slenderness ratio \( \frac{R_e}{\bar{f}} \) is greater than about 100. Gengshu, Pi et al. (2008) also find that the axial force is close to the nominal value for an arch with \( \phi_0 \leq 45^\circ \) (the subtended angle of the arch is less than or equal to 90°) and can thus be considered a “deep” arch.

![Figure 6 Distribution of Axial Force through Arch for a Slenderness of \( \frac{R_e}{\bar{f}} = 60 \)](image)

Figure 6 Distribution of Axial Force through Arch for a Slenderness of \( \frac{R_e}{\bar{f}} = 60 \)
Figure 7 Ratio of Maximum Axial Force to Nominal Axial Force versus Slenderness for $EA/GA_s = 25$ (Pinned Supports)

Figure 8 Ratio of Maximum Axial Force to Nominal Axial Force versus Slenderness for $EA/GA_s = 3$ (Pinned Supports)
Figure 9 Ratio of Maximum Axial Force to Nominal Axial Force versus Slenderness for $\frac{EA}{GA} = 25$ (Fixed Supports)

Figure 10 Ratio of Maximum Axial Force to Nominal Axial Force versus Slenderness for $\frac{EA}{GA} = 3$ (Fixed Supports)
5 Buckling of a Ring under Hydrostatic Pressure

Consider the case of a circular ring under a hydrostatic pressure loading. The hydrostatic load remains normal to the centroidal axis and is proportional to the deformed stretch of the centroidal axis, hence the components in the radial and tangential direction of loading are given by:

\[ p^\prime R_o = -pR_o \lambda_{\phi_o} \sin(\theta + \phi_o) \]
\[ p^\prime R_o = -pR_o \lambda_{\phi_o} \cos(\theta + \phi_o) \]

As pointed out in Oran and Reagan (1969), Simitses (1976), Oran and Bayazid (1978), Hodges (1999), Simitses and Hodges (2006), the hydrostatic loading on the ring is locally non-conservative. However, the whole external hydrostatic loading system is conservative. We now substitute this loading state into the equilibrium equations (49) to (51), to obtain:

\[ N \left( 1 - \frac{d\theta}{d\phi} \right) \frac{dQ}{d\phi} = -pR_o \lambda_{\phi_o} \cos(\phi_o) \]

\[ Q \left( 1 - \frac{d\theta}{d\phi} \right) \frac{dN}{d\phi} = -pR_o \lambda_{\phi_o} \sin(\phi_o) \]

\[ \frac{dM}{ds} = -\lambda_{\phi_o} Q = N \lambda_{\phi_o} \sin(\phi_o) - Q \lambda_{\phi_o} \cos(\phi_o) \]

Prior to buckling the ring has an axial force equal to \( N_o \) and no bending or shear. The initial axial force is related to the hydrostatic pressure by the following:
\[ N_o = -pR_o \bar{\lambda}_{\phi o} \quad \bar{\lambda}_{\phi o} = 1 + \frac{N_o}{EA} \quad \therefore pR_o = \frac{N_o}{1 + \frac{N_o}{EA}} \] (72)

At the bifurcation state, we consider variation about the equilibrium state. The variational form of the three equilibrium equations (69) to (71), are therefore:

\[ \delta N = N_o R_o \left( \frac{\delta M}{E I} \right) - \frac{d \delta Q}{d \phi} = -pR_o \frac{\delta N}{EA} = \frac{N_o}{1 + \frac{N_o}{EA}} \] (73)

\[ \therefore \delta N = \left[ N_o R_o \left( \frac{\delta M}{E I} \right) + \frac{d \delta Q}{d \phi} \right] \left( 1 + \frac{N_o}{EA} \right) \]

\[ \delta Q + \frac{d \delta N}{d \phi} = pR_o \bar{\lambda}_{\phi o} \delta \phi_o = -pR_o \frac{\delta Q}{GA_s} = \frac{N_o}{1 + \frac{N_o}{EA}} \] (74)

\[ \therefore \left( 1 - \frac{N_o}{GA_s} \right) \delta Q + \frac{d \delta N}{d \phi} = 0 \]

\[ \frac{d \delta M}{d \phi} = R_o \left( N_o \bar{\lambda}_{\phi o} \delta \phi_o - \delta Q \bar{\lambda}_{\phi o} \right) = R_o \left( \frac{N_o}{GA_s} \left[ 1 + \frac{N_o}{EA} \right] \right) \delta Q \] (75)

Differentiating Eqn (73) and making use of (74) and (75) we can derive a differential equation in the one variable \( \delta Q \) thus:

\[ b \delta Q + \frac{d^2 \delta Q}{d \phi^2} = 0 \] (76)

where the term \( b \) is defined by:

\[ b = \left( 1 - \frac{N_o}{GA_s} \right) + \frac{N_o R_o^2}{E I} \left( \frac{N_o}{GA_s} \left[ 1 + \frac{N_o}{EA} \right] \right) \]

\[ \left( 1 + \frac{N_o}{EA} \right) \] (77)

The solution to the second order differential equation is simply:

\[ \delta Q = C_1 \sin(\sqrt{b} \phi) + C_2 \cos(\sqrt{b} \phi) \] (78)
If we assume from symmetry that the shear $\delta Q$ is equal at $\phi = 0 \& \pi$ then we have the condition:

$$\sin(\sqrt{b}\pi) = 0 \quad \cos(\sqrt{b}\pi) = 1 \quad \therefore \sqrt{b} = 2n \quad n = 1, 2, 3...$$ \quad (79)

which when using Eqn (77) leads to a cubic equation for the buckling load. If we ignore the prebuckling axial displacements $\left(\frac{N_0}{EA} \square 1\right)$ then $b$ would simplify and the buckling load is the solution to a quadratic, thus:

$$b \approx \left(1 - \frac{N_0}{GA_1}\right)\left(1 - \frac{N_0}{EA} \frac{R_0^2}{\bar{r}^2}\right) = 4n^2$$ \quad (80)

$$\therefore \frac{N_{eq}}{EA} = -\frac{p_0 R_0}{EA} \approx \frac{1}{2} \left[ \frac{GA_1}{EA} + \frac{1}{E \left(\frac{R_0}{\bar{r}}\right)^2} \left( \frac{GA_1}{EA} - \frac{1}{E \left(\frac{R_0}{\bar{r}}\right)^2} \right)^2 + 16n^2 \frac{1}{E \left(\frac{R_0}{\bar{r}}\right)^2} \frac{GA_1}{EA} \right]$$

The limit as $\frac{R_0}{\bar{r}} \rightarrow 0$

$$\frac{p_0 R_0}{EA} \rightarrow (-1 + 4n^2) \frac{E \bar{A}}{GA_1}$$ \quad (81)

If we further ignore the shear deformations $\left(\frac{N_0}{GA_1} \square 1\right)$, the solution reduces to the classical buckling formula for a circular ring under hydrostatic pressure, Timoshenko and Gere (1963), that is:

$$b \approx 1 - \frac{N_0 R_0^2}{EI} = 4n^2 \quad \frac{N_{eq}}{EA} = -\frac{p_0 R_0}{EA} = \frac{(4n^2 - 1) E \bar{I}}{E A R_0^2} = \left(\frac{R_0}{\bar{r}}\right)^2$$ \quad (82)
Figure 12 depicts the buckling load solution based on Eqns (77) and (79), as a function of the ring slenderness for two representative axial to shear rigidity ratios $\frac{EA}{GA} = 3$ or 25. Also shown is the classical solution Eqn (82). As with most buckling problems where shear is considered, the effects of shear rigidity are evident only at very small slenderness. Figure 12 and the numerical solution indicate that for a $\frac{EA}{GA} = 25$ there is less than 10% difference between the classical solution and the solution derived here for a slenderness greater than 25. At a slenderness of 10, the percentage difference is close to 35%.

6 Buckling of Circular High Arches under Radially Directed Dead Pressure

Consider the prismatic circular arch shown in Figure 5 under a radially applied conservative distributed load which induces an initial axial compressive load. As discussed above, the pre-buckling deformations produce no bending or shear. To ascertain the buckling load, we apply small kinematically admissible variations denoted by the symbol $\delta$ of the displacement field. Initially for the arch before perturbations, we have for the initial axial force and associated stretch:
\[ N_o = -p_o R_o \bar{\lambda}_{\phi_o} = 1 + \frac{N_o}{EA} \]  \hspace{1cm} (83)

The boundary conditions for this problem (see Simitses (1976)) are that:

at \( \phi = \phi_o \) & \( \pi - \phi_o \) \hspace{1cm} \delta u_o = 0 \hspace{1cm} \delta v = 0 \hspace{1cm} \delta N = 0 \hspace{1cm} \delta M = 0 \hspace{1cm} (84)

Taking variations of the equilibrium equations gives

\[ \delta N = E\bar{A}\delta \lambda_{\phi_o} = -p_o R_o^2 \left( \frac{\delta M}{EI} \right) + \frac{d\delta Q}{d\phi} \]  \hspace{1cm} (85)

\[ \delta Q = G\bar{A}\bar{\lambda}_{\phi_o} \delta \phi_o = p_o R_o \delta \theta - \frac{d\delta N}{d\phi} \]  \hspace{1cm} (86)

\[ \frac{d\delta M}{d\phi} = \frac{E \bar{I}}{R_o} \frac{d^3 \delta \theta}{d\phi^2} = R_o \left( N_o \bar{\lambda}_{\phi_o} \delta \phi_o - \delta Q \bar{\lambda}_{\phi_o} \right) = b \delta Q \]  \hspace{1cm} (87)

where \( b = -R_o \left( \frac{p_o R_o}{GA} + \left[ 1 - \frac{p_o R_o}{EA} \right] \right) \)

Differentiating Eqn (86), using Eqns (37) and adding this to Eqn (85) gives:

\[ \delta N + \frac{d^2 \delta N}{d\phi^2} = 0 \hspace{1cm} \therefore \hspace{1cm} \delta N = C_3 \sin(\phi) + C_4 \cos(\phi) \]  \hspace{1cm} (88)

For an arch with pinned supports at the base, the boundary conditions defined in Eqn (84) gives \( \delta N = 0 \) at the supports, and therefore

\[ \delta N = 0 \hspace{1cm} \therefore \hspace{1cm} C_3 = C_4 = 0 \]  \hspace{1cm} (89)

Substituting the condition that \( \delta N = 0 \) into Eqn (85), differentiating and using Eqns (87), we derive a differential equation for \( \delta Q \) thus:

\[ a \delta Q + \frac{d^2 \delta Q}{d\phi^2} = 0 \hspace{1cm} a = -\frac{p_o R_o^2}{EI} b \]  \hspace{1cm} (90)

The solution to the above second order differential equation is then:

\[ \delta Q = G\bar{A}\bar{\lambda}_{\phi_o} \delta \phi_o = C_3 \sin(\sqrt{a} [\phi - \phi_o]) + C_4 \cos(\sqrt{a} [\phi - \phi_o]) \]  \hspace{1cm} (91)

and hence
\[
\delta M = \frac{E \bar{I}}{R_o} d\phi \frac{d\theta}{d\phi} = \frac{b}{\sqrt{a}} \left\{-C_1 \cos \left(\sqrt{a} [\phi - \phi_o] \right) + C_2 \sin \left(\sqrt{a} [\phi - \phi_o] \right) \right\} 
\]

(92)

For an arch with pinned supports, the bending moments at \( \phi = \phi_o \) & \( \pi - \phi_o \) are zero, therefore:

\[
C_1 = 0 \quad C_2 \sin \left(\sqrt{a} [\pi - \phi_o] \right) = 0 \quad \therefore \sin \left(\sqrt{a} [\pi - 2\phi_o] \right) = 0
\]

(93)

The first variation of the stretch \( \delta \lambda_{\phi_o} \) can be obtained from Eqns (18), therefore

\[
\frac{\delta N}{EA} = \delta \lambda_{\phi_o} = \frac{\delta u_{o,\phi} + \delta v}{R_o} = 0 \quad \therefore \delta u_{o,\phi} = -\delta v
\]

(94)

The first variation of rotation terms \( \delta \theta + \delta \phi_o \) is derived from Eqn (23), therefore, using Eqn (94) we have

\[
\delta (\theta + \phi_o) = \delta \lambda_{\phi_o} \quad \therefore \delta \left(\frac{d\theta}{d\phi} + \frac{d\phi_o}{d\phi} \right) = \delta \lambda_{\phi_o} + \delta v
\]

(95)

Equations (94) \& (95) can be solved using Eqns (92) \& (93) and applying the boundary conditions \( \delta v = 0 \) at \( \phi = \phi_o \) & \( \pi - \phi_o \)and \( \delta u = 0 \) at \( \phi = \phi_o \), hence

\[
\delta v = D \sin \left(\sqrt{a} [\phi - \phi_o] \right) \quad \delta u_o = \frac{D}{\sqrt{a}} \left\{1 - \cos \left(\sqrt{a} [\phi - \phi_o] \right) \right\}
\]

(96)

where \( D \) is a constant of integration. Applying the boundary condition \( \delta u = 0 \) at \( \pi - \phi_o \) and recalling Eqn (93), gives:

\[
\cos \left(\sqrt{a} [\pi - 2\phi_o] \right) = 1 \quad \sin \left(\sqrt{a} [\pi - 2\phi_o] \right) = 0
\]

(97)

Therefore:

\[
\sqrt{a} [\pi - 2\phi_o] = 2n\pi \quad n = 1, 2, 3,\ldots
\]

(98)

where \( n \) is an integer (representing the buckling mode number). The critical buckling load is then the solution to the following quadratic:

\[
\frac{p_o R_o^2}{EA} \left[ \frac{p_o R_o}{GA} + \left(1 - \frac{p_o R_o}{EA} \right) \right] = \frac{n^2 \pi^2}{\left[ \frac{\pi}{2} - \phi_o \right]^2}
\]

(99)

If we ignore both the axial deformation prior to buckling and shear deformation during buckling then Eqn (99) reduces to
\[
\frac{p_c R_0^3}{EI} = \frac{n^2 \pi^2}{\left(\frac{\pi}{2} - \phi_0\right)^2} 
\]

(100)

This solution agrees with Simitses (1976), Hodges (1999), Simitses and Hodges (2006), Gengshu, Pi et al. (2008). This solution for the buckling load is not the same as the buckling formula given in Vlasov (1959), Timoshenko and Gere (1963), Pi, Bradford et al. (2002) where the buckling load is quoted as:

\[
\frac{p_c R_0^3}{EI} = \frac{n^2 \pi^2}{\left[\frac{\pi}{2} - \phi_0\right]^2 - 1} 
\]

(101)

Gengshu, Pi et al. (2008) discuss the various buckling formulas derived for the problem of a uniform radial load and attribute the differences to the accuracy of the definition of finite strain. The problem with the solution provided by Timoshenko and Gere (1963) is that it is based on taking the bending moment on any section of the arch due to the reaction and distributed radial load as:

\[
M = p_c R_0 v(\phi) 
\]

(102)

It is shown in Appendix A, however that the correct expression for the bending moment should be:

\[
M = p_c R_0 \left[ v(\phi) - \int_0^{\phi} u_\phi(\bar{\phi}) d\bar{\phi} \right] 
\]

(103)

Using this expression for the bending moment gives rise to the solution given in Eqn (100).

There are very few experimental results available for the radial loading problem. The reference Wilson, Holloway et al. (1971) contains one experimental set of load deflection results for the pinned ended column under radial loading. Wilson, Holloway et al. (1971) tested a rectangular Plexiglas specimen arch with an included angle of 120° and a radius of 2.875 inches (73.025 mm). The results are plotted in Figure 13. Although their results do not confirm either solution, Eqns (100) or (101), their results had a peak load at least 30% above that predicted by Eqn (101). This higher peak load was attributed to friction in the supports.

Figure 14 shows a Southwell plot using the results of Wilson, Holloway et al. (1971) giving an estimated buckling load per arch length of 10.9 lbs/in while Eqn (100) gives 9.125 lbs/in and Eqn (101) gives 8.11 lbs/in.
Figure 13 Experimental Results from Wilson, Holloway et al. (1971)

Figure 14 Southwell Plot for Experimental Results from Wilson, Holloway et al. (1971)

The solution to the quadratic equation (99) for buckling load, which incorporates the axial deformation prior to buckling and shear deformation during buckling, is then:
\[
\frac{E\tilde{A}}{GA_i} = 1 \quad \frac{p_e R_0}{EA} = \frac{1}{2 \left( 1 - \frac{E\tilde{A}}{GA_i} \right)} \left[ 1 \pm \sqrt{1 - \frac{4n^2\pi^2}{\left( \frac{\pi}{2} - \phi_0 \right)^2 \left( \frac{R_0}{\tilde{r}} \right)^2 \left( 1 - \frac{E\tilde{A}}{GA_i} \right)} - \left( \frac{R_0}{\tilde{r}} \right)^2 \left( 1 - \frac{E\tilde{A}}{GA_i} \right) } \right]
\]

(104)

In which \( \nu \) is the Poisson's ratio. The solution with the minus sign holds. Equation (104) has an identical form to the buckling formula derived in Attard (2003), Attard and Hunt (2008) for a straight column under axial loading \( \left( \frac{E\tilde{A}}{GA_i} \neq 1 \right) \), written here:

\[
\frac{P_e}{EA} = \frac{1}{2 \left( 1 - \frac{E\tilde{A}}{GA_i} \right)} \left[ 1 \pm \sqrt{1 - \frac{4n^2\pi^2}{\left( \frac{L}{r} \right)^2 \left( 1 - \frac{E\tilde{A}}{GA_i} \right)} } \right]
\]

(105)

In the above \( P_e \) is the column buckling load and \( L/r \) is the column slenderness. Comparing Eqns (104) with (105), we see that the arch under dead load and the column under axial force have the same buckling formula with an equivalent arch slenderness of:

\[
\frac{L}{r} = \left[ \frac{\pi}{2} - \phi_0 \right] \left( \frac{R_0}{\tilde{r}} \right)
\]

(106)

Real solutions to the buckling load exist for

\[
\left[ \frac{\pi}{2} - \phi_0 \right]^2 \left( \frac{R_0}{\tilde{r}} \right)^2 \geq 4n^2\pi^2 \left( 1 - \frac{E\tilde{A}}{GA_i} \right)
\]

(107)

This will always be satisfied for any slenderness since the elastic axial rigidity will always be greater than the shear rigidity. Figure 15 and Figure 16 show the buckling solution based on Eqn (104) as a function of the slenderness for two representative axial to shear rigidity ratios \( \frac{E\tilde{A}}{GA} = 3 \) or 25 for various values of the initial arch angle \( \phi_0 = 0, 45^\circ, 70^\circ \) & 80°. The buckling capacity based on Eqn (104) increases with the initial angle and the axial to shear rigidity ratio.
Figure 15 Buckling versus Radial Slenderness for an Arch under Radially Directed Dead Pressure, having $EA/GA_i=3$

Figure 16 Buckling versus Radial Slenderness for an Arch under Radially Directed Dead Pressure, having $EA/GA_i=25$
7 Solution using Second Variation of Total Potential

The problem of the radially applied distributed load can also be solved by considering the second variation of the total potential energy. The second variation of the total potential is derived from Eqn (43), that is:

$$\delta W = \int_{0}^{L} \left[ N \delta \lambda_{n0} + Q \delta \lambda_{n0} + M \delta \theta_{\delta} + p_r \delta v_{\delta} \right] ds = 0$$

$$\therefore \delta^2 W = \int_{0}^{L} \left[ \frac{1}{2} \delta^{2} N \delta \lambda_{n0} + \frac{1}{2} \delta Q \delta \lambda_{n0} + \frac{1}{2} \delta M \delta \theta_{\delta} \right] ds$$

where $N, Q, M$ are the axial force, shear and bending moment, respectively, prior to buckling. If we assume $N = -p_r R_o$ and there is no shear and bending moment prior to buckling, the second variation of potential is:

$$\delta^2 W = \int_{0}^{L} \left[ \frac{1}{2} E \lambda \left( \delta \mu_{n0} \right)^2 + \frac{1}{2} G \lambda \left( \delta \varphi_{\delta} \right)^2 + \frac{1}{2} E \lambda \left( \delta \theta_{\delta} \right)^2 \right] ds$$

(108)

The details of the derivation of Eqn (109) are contained in Appendix B. Expanding by parts, ignoring the axial term, the following equations are derived which represent equilibrium in the perturbed state:

$$E \lambda \delta \theta_{\delta} + p_r R_o \left( \delta \theta + \delta \varphi_{\delta} \right) = 0$$

(110)

$$G \lambda \delta \varphi_{\delta} - p_r R_o \delta \theta = 0 \therefore \delta \varphi_{\delta} = \frac{p_r R_o}{G \lambda} \delta \theta$$

(111)

Combining Eqns (110) and (111) gives the second order differential equation:

$$E \lambda \delta \theta_{\delta} + p_r R_o \left( 1 + \frac{p_r R_o}{G \lambda} \right) \delta \theta = 0$$

(112)

which can be rewritten in the form:

$$\bar{p} \delta \theta_{\delta} + b \delta \theta = 0 \quad b = \frac{p_r R_o}{E \lambda} \left( 1 + \frac{p_r R_o}{G \lambda} \right)$$

(113)

This equation is the same as the classical equation for the buckling of a straight column under axial force ignoring the axial deformation prior to buckling but including shear deformation during buckling, as shown in Attard (2003), Attard and Hunt (2008). The equivalent column slenderness was derived in Eqn (106).


8 Summary

A formulation for finite strain buckling analysis of high circular arches that includes the effects of shear deformation is presented in the current study. Cases where shear deformations gain importance are identified. The kinematics are developed using the Timoshenko beam analogy. The normal and shear finite strain expressions are derived based on the normal and shear components of the relative stretch of the longitudinal fibres. Constitutive relations for stresses and internal actions are derived using hyperelastic constitutive modeling for a compressible isotropic neo-Hookean material proposed in Attard and Hunt (2004). Virtual work and equilibrium equations for in-plane behavior of circular arches under applied conservative tractions are presented. The prebuckling linearized solution for a prismatic circular arch under a radially distributed conservative load is derived. The axial force in the arch was shown to be approximately equal to the nominal value and the arch considered a “deep” arch when the subtended angle of the arch is less than or equal to 90° and the slenderness ratio is greater than about 60 for pin supports and greater than about 100 for an arch with fixed supports. The closed form solution for the buckling of a ring under hydrostatic pressure including the effects of shear was derived. The closed form solution for the buckling of a circular arch under radially directed conservative distributed load, was also derived. The equations developed in Simitses (1976), Hodges (1999), Simitses and Hodges (2006), Gengshu, Pi et al. (2008) agree with the expressions derived here if both the axial and shear deformations are ignored. The solution for the buckling load derived here is not the same as the buckling formula given in Vlasov (1959), Timoshenko and Gere (1963), Pi, Bradford et al. (2002). The reason for the discrepancy revolves around the correct expression for the bending moment at any section due to the axial force and lateral displacement. Experimental results in Wilson, Holloway et al. (1971) tended to support the solution provided here. Consistent expressions for the second variation of the total potential are also given. The significance of shear deformations are important for low relative shear rigidity, low slenderness and increasing subtended angle.
9 References


Figure 17 Freebody of Segment of an Arch under Radial Distributed Load.

\[
M = \int_0^\phi \left[ p_o R_o \cos \tilde{\phi} \left\{ R_o \left( \sin \phi - \sin \tilde{\phi} \right) + \left( u_o (\phi) \cos \phi + v(\phi) \sin \phi \right) \right\} + \left( - u_o (\phi) \cos \phi + v(\phi) \sin \phi \right) \right] d\tilde{\phi} \\
+ p_o R_o \sin \phi \left\{ R_o \left( \cos \tilde{\phi} - \cos \phi \right) + \left( - u_o (\phi) \sin \phi + v(\phi) \cos \phi \right) \right\} \\
+ p_o R_o \left[ \cos \tilde{\phi} - 1 - u_o (\phi) \sin \phi + v(\phi) \cos \phi \right] \\
= p_o R_o \left[ v(\phi) - \int_0^\phi u_o (\tilde{\phi}) d\tilde{\phi} \right] \\
\therefore \frac{dM}{ds} = p_o R_o \tilde{v}_{,\phi}
\]
Appendix B – First and Second Variation Terms

Assuming $\ddot{u}_{0,s} << 1$, $\ddot{v}_{0,s} << 1$, $\lambda_{0o} << 1$, $\sin \varphi_o \approx \varphi_o$ and $\cos \varphi_o \approx 1$ the first and second variation of the normal and shear components of stretch, and the bending angle are derived as:

$$\delta\lambda_{0o} = \delta \ddot{u}_{0,s} + \ddot{v}_{0,s} \delta \ddot{v}_{0,s} - \varphi_o \delta \varphi_o$$  \hspace{1cm} (115)

$$\delta\lambda_{0p} = \varphi_o \left( \delta \ddot{u}_{0,s} + \ddot{v}_{0,s} \delta \ddot{v}_{0,s} \right) + \delta \varphi_o$$  \hspace{1cm} (116)

$$\delta\theta_s = \left( 2 \ddot{v}_{0,s} \ddot{u}_{0,s} - \ddot{v}_{0,s} \right) \delta \ddot{u}_{0,s} + \delta \ddot{v}_{0,s} - \left( 2 \ddot{v}_{0,s} \ddot{v}_{0,s} + \ddot{u}_{0,s} \right) \delta \ddot{v}_{0,s}$$  \hspace{1cm} (117)

$$- \dddot{v}_{0,s} \delta \ddot{u}_{0,s} - \delta \varphi_o$$

$$\delta^2 \lambda_{0o} = \frac{1}{2} \left( \dddot{v}_{0,s} \delta \ddot{u}_{0,s} - \delta \dddot{v}_{0,s} \right)^2 - \frac{1}{2} \left( \delta \varphi_o \right)^2 - \varphi_o \delta \varphi_o \left( \delta \ddot{u}_{0,s} + \ddot{v}_{0,s} \delta \ddot{v}_{0,s} \right)$$  \hspace{1cm} (119)

$$\delta^2 \lambda_{0p} = \frac{1}{2} \varphi_o \left( \dddot{v}_{0,s} \delta \ddot{u}_{0,s} - \delta \dddot{v}_{0,s} \right)^2 - \frac{1}{2} \varphi_o \left( \delta \varphi_o \right)^2 + \delta \varphi_o \left( \delta \ddot{u}_{0,s} + \ddot{v}_{0,s} \delta \ddot{v}_{0,s} \right)$$  \hspace{1cm} (120)

If we ignore any prebuckling deformations then Eqns (115) to (120) simplify to:

$$\delta\lambda_{0o} = \delta \ddot{u}_{0,s}, \ \delta\lambda_{0p} = \delta \varphi_o, \ \delta\theta_s = \delta \ddot{v}_{0,s} - \delta \varphi_o$$  \hspace{1cm} (121)

$$\delta^2 \lambda_{0o} = \frac{1}{2} \left( \dddot{v}_{0,s} \right)^2 - \frac{1}{2} \left( \delta \varphi_o \right)^2 = \frac{1}{2} \left( \delta \theta + \delta \varphi_o \right)^2 - \frac{1}{2} \left( \delta \varphi_o \right)^2$$  \hspace{1cm} (122)

Substituting Eqns (121) and (122) into Eqn (108), using the constitutive relationships Eqn (37), taking $N_o = -p_o R_o$ and assuming there is no shear or bending moment prior to buckling gives for the second variation of total potential:

$$\delta^2 W = \int_0^l \left[ \frac{1}{2} EA \left( \delta \ddot{u}_{0,s} \right)^2 + \frac{1}{2} GA \left( \delta \varphi_o \right)^2 + \frac{1}{2} EI \left( \delta \theta_s \right)^2 \right] ds$$  \hspace{1cm} (123)