BEHAVIOUR OF STEEL FIBRE REINFORCED MORTAR IN SHEAR III: VARIABLE ENGAGEMENT MODEL II

BY

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Behaviour of Steel Fibre Reinforced Mortar in Shear III: Variable Engagement Model II

Gregory G Lee and Stephen J Foster

Fibres, reinforced mortar, orientation angle, shear stress, pullout, fracture, variable engagement model, snubbing, hook, displacement, orientation factors, hybrid combinations.

In this report, a model designated the Variable Engagement Model II (VEMII) is developed to describe the behaviour of randomly oriented discontinuous steel fibre reinforced composites loaded in shear. The model is derived from the experimental work carried out on the direct shear behaviour of discrete hooked-ended and straight steel fibres and verified from a series of randomly distributed fibre reinforced mortar specimens, reported earlier. Two forms of models are analysed: 1) a model based on the observation of concentrated shear stresses at the fibre hook and in the snubbing zone; and 2) a uniform fibre bond stress applied along the embedded part of the fibre. The concentrated bond stress approach and the uniform approach were found to give reasonable comparisons with the test data for the hooked-ended fibres but were conservative for the straight fibres. The VEMII model provides a versatile approach that can also be applied to hybrid fibre combinations.

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NOMENCLATURE

$A_c$  cross-sectional area of matrix at the shear plane
$A_f$  cross-sectional area of a fibre
$a_c$  concrete coefficient for plain concrete
$a_e$  resultant elastic stress-strain constant
$a_{ep}$ resultant elastic-plastic stress-strain constant in tension region
$a_{ep1}$ resultant elastic-plastic stress-strain constant in compression region
$a_b$  proportion of hook zone engaged
$a_p$  resultant plastic stress-strain constant
$a_{snub}$ proportion of snubbing zone engaged
$a_w$  displacement coefficient for plain concrete
$a_{yu}$ ratio between elastic yield strain and ultimate strain
$C_e$  elastic compression force in fibre due to bending
$C_{ep}$ resultant elastic compression force in fibre due to bending (elastic-plastic model)
$C_{ep1}$ resultant plastic compression force in fibre due to bending (elastic-plastic model)
$C_p$  plastic compression force in fibre due to bending
$c$  axial fracture ratio
$d_f$  fibre diameter
$d_0$ distance between the neutral axis and fibre plastic centroid
$d_i$ distance between the tension yield strain and fibre plastic centroid
$d_2$ distance between the compression yield strain and fibre plastic centroid
$\bar{d}$ bending moment factor expressed as the ratio between $d_0$ and $d_f/2$
$\bar{d}_i$ tension yield strain factor expressed as the ratio between $d_i$ and $d_f/2$
$\bar{d}_2$ compression yield strain factor expressed as the ratio between $d_2$ and $d_f/2$
$E_f$ fibre modulus of elasticity
$F$  end effect factor
$f_{cm}$ the mean value of the compressive strength of concrete or mortar at the relevant age
$f_{ct}$ the mean value of the tensile strength of concrete or mortar at the relevant age
$f_{ct, av}$ the average tensile strength of concrete or mortar at the relevant age
$G_{FS}$ direct shear fibre fracture energy
$H$ hooked-ended steel fibres
$K_r$ direct shear global orientation factor
$k$ fibre engagement factor; or direct shear local orientation factor
$k_{ave}$ average direct shear local orientation factor
$k_i$ local orientation factor for the $i$th fibre
$l_a$ initial fibre embedment length
$l_{br}$ total length of bridging zone in direct shear
$l_c$ critical fibre length
$l_e$ fibre embedment length beyond the shear plane
$l_f$ total fibre length
$l_{hook}$ total length of hook in direct shear
$l_{snub}$ total length of snubbing zone in direct shear
$l_{st}$ total length of straight zone in direct shear
$l_{sv}$ vertical length of snubbing zone in direct shear
$M_e$ elastic fibre bending moment
$M_{ep}$ elastic-plastic fibre bending moment
$M_p$ plastic fibre bending moment
$M_{snub}$ snubbing moment
$N$ total number of fibres
$NA$ position of neutral axis
$N_{al}$ number of aligned fibres
$N_e$ resultant elastic tensile force in fibre
$N_{ep}$ resultant elastic-plastic tensile force in fibre
$N_p$ resultant plastic tensile force in fibre
$n$ number of fibres engaged
$P_f$ shear load per fibre
$P_f^\text{fract}$ shear load per fibre in the overlapping hook and snubbing zones
$P_f^\text{frac}$ fibre fracture load
$P_{f0}$ load per fibre at peak
$P_{f0,av}$ average peak load per fibre
$P_{hook}$ load per fibre due to end hook
$P_{snub}$ load per fibre due to snubbing
$p$ proportion of fibres remaining embedded across a crack for a given vertical displacement
$R$ radius of fibre bend in snubbing zone; or end reduction factor
$S$ straight steel fibres
$T_{ax}$ elastic axial tension force in fibre
\( T_{\text{epf}} \) elastic-plastic axial tension force in fibre
\( T_r \) resultant elastic tension force in fibre due to bending
\( T_{ep} \) resultant elastic tension force in fibre due to bending (elastic-plastic model)
\( T_{ep1} \) resultant plastic tension force in fibre due to bending (elastic-plastic model)
\( T_p \) resultant plastic tension force in fibre due to bending
\( V_c \) shear strength of plain concrete
\( V_{\text{conc}} \) volume of concrete
\( w_{es} \) vertical displacement at the point of engagement of a fibre in direct shear
\( w_s \) crack sliding displacement
\( w_v \) vertical displacement of a fibre in direct shear or measured displacement
\( w_{v \text{ peak}} \) measured vertical displacement at peak shear stress
\( w_{v_l} \) vertical displacement on the elastic curve corresponding to the point \( \tau_l \)
\( w_w \) crack width
\( Z_f \) fibre section modulus
\( z \) distance from fibre centreline, positive above the fibre centreline and negative below the fibre centreline
\( z_{ce} \) distance from fibre centreline to centroid of elastic compression force
\( z_{cep} \) distance from fibre centreline to centroid of elastic compression force (elastic-plastic model)
\( z_{ep1} \) distance from fibre centreline to centroid of plastic compression force (elastic-plastic model)
\( z_{ep} \) distance from fibre centreline to centroid of plastic compression force
\( z_{te} \) distance from fibre centreline to centroid of elastic tension force
\( z_{tep} \) distance from fibre centreline to centroid of elastic tension force (elastic-plastic model)
\( z_{tep1} \) distance from fibre centreline to centroid of plastic tension force (elastic-plastic model)
\( z_{tp} \) distance from fibre centreline to centroid of plastic tension force
\( \alpha_{I1} \) direct shear engagement constant
\( \alpha_f \) aspect ratio of a fibre
\( \alpha_{\text{snub}} \) enclosed angle of fibre bend in snubbing zone
\( \beta \) fibre fracture bending coefficient
\( \varepsilon_{ae} \) elastic axial strain
\( \varepsilon_{aep} \) elastic-plastic axial strain
\( \varepsilon_{up} \) plastic axial strain
\( \varepsilon_b \) bending strain

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\( \varepsilon_{bc} \)  
esthetic bending strain
\( \varepsilon_{bep} \)  
esthetic-plastic bending strain
\( \varepsilon_{pp} \)  
esthetic bending strain
\( \varepsilon_{ce} \)  
esthetic elastic compressive strain in fibre
\( \varepsilon_{cep} \)  
esthetic elastic-plastic compressive strain in fibre
\( \varepsilon_{cp} \)  
esthetic plastic compressive strain in fibre
\( \varepsilon_{fu} \)  
esthetic ultimate tensile strain of a steel fibre
\( \varepsilon_{fy} \)  
esthetic yield strain of a fibre
\( \varepsilon_{te} \)  
esthetic resultant elastic tensile strain in fibre
\( \varepsilon_{tep} \)  
esthetic resultant elastic-plastic tensile strain in fibre
\( \phi \)  
esthetic angle between the crack surface and fibre
\( \kappa \)  
esthetic curvature of fibre
\( \theta \)  
esthetic steel fibre angle of orientation
\( \theta_{crit} \)  
esthetic critical angle of orientation for fibre engagement in direct shear
\( \theta_{lim} \)  
esthetic limiting angle of orientation for fibre engagement in direct shear
\( \rho_f \)  
esthetic volume fraction of fibres
\( \sigma_{ue} \)  
esthetic elastic axial stress
\( \sigma_{uep} \)  
esthetic elastic-plastic axial stress
\( \sigma_{ue} \)  
esthetic elastic bending stress
\( \sigma_{uep} \)  
esthetic elastic-plastic bending stress
\( \sigma_{ce} \)  
esthetic resultant elastic compressive stress in fibre
\( \sigma_{cep} \)  
esthetic resultant elastic-plastic compressive stress in fibre
\( \sigma_{fu} \)  
esthetic ultimate tensile strength of a steel fibre
\( \sigma_{f0}(\theta) \)  
esthetic fracture strength for a fibre oriented at an angle \( \theta \) to the shear plane
\( \sigma_{f0} \)  
esthetic fibre strength for zero bending stress
\( \sigma_{fy} \)  
esthetic yield tensile strength of a steel fibre
\( \sigma_n \)  
esthetic normal stress
\( \sigma_{ie} \)  
esthetic resultant elastic tensile stress in fibre
\( \sigma_{iwp} \)  
esthetic resultant elastic-plastic tensile stress in fibre
\( \sigma_{ce} \)  
esthetic elastic stress in fibre at position \( z \)
\( \sigma_{cep} \)  
esthetic elastic compressive stress in fibre at position \( z \) (elastic-plastic model)
\( \sigma_{rept} \)  
esthetic elastic tension stress in fibre at position \( z \) (elastic-plastic model)
\( \tau \)  
esthetic shear stress
\( \sigma_{fu} \)  
esthetic effective ultimate tensile strength of the fibre
\( \tau_{ave} \)  
esthetic average bond stress along the fibre length
\( \tau_p \)  
esthetic shear stress from concrete matrix contribution
\( \tau_{peak} \)  
esthetic peak reinforced matrix strength
\[ \tau_f \quad \text{shear stress from fibre contribution} \]
\[ \tau_{hook} \quad \text{shear stress along the embedded hook fibre end under direct shear} \]
\[ \tau_{\max} \quad \text{ultimate shear stress} \]
\[ \tau_{\text{peak}} \quad \text{peak shear stress} \]
\[ \tau_{\text{snub}} \quad \text{shear stress along the fibre length in the snubbing zone under direct shear} \]
\[ \tau_{\text{snub,ave}} \quad \text{average snubbing shear stress for all engaged fibres} \]
\[ \tau_{st} \quad \text{shear stress along the embedded straight fibre lengths under direct shear} \]
\[ \tau_P \quad \text{shear stress at } w_v \]
\[ \tau_I \quad \text{elastic portion of shear stress at } w_{vl} \]
CHAPTER 1

INTRODUCTION

This report is the final of three reports related to the direct shear behaviour of steel fibre reinforced mortar. The previous reports dealt with individual and volume concentration steel fibre specimens and gamma ray imaging tests carried out at the University of New South Wales Concrete Testing Laboratory to examine the direct shear performance of steel fibre reinforced concrete. This report concentrates on the development of the variable engagement model designated, VEMII.

1.1 General

The investigation carried out by Voo and Foster (2003) into the uniaxial tensile behaviour of steel fibres introduced the expression Variable Engagement Model (VEM). The Variable Engagement Model describes the performance of randomly oriented discontinuous fibres under the influence of uniaxial tension. A similar broad approach is used in the development of the model for steel fibres subjected to direct shear, designated the Variable Engagement Model II (VEMII).

The direct shear model proposed in this report draws on the experimental results and observations arising from the two companion reports by Lee and Foster (2006a, 2006b). The proposed models from previous researchers are examined in the next section.

1.2 Previous Direct Shear Models

In the report by Lee and Foster (2006a) investigations were carried out into the behaviour of randomly distributed steel fibres subjected to direct shear. It was noted in the report that previous research in the field of direct shear did not match the same level of scrutiny as its tensile behaviour. As a consequence, there is not a great deal of data available. Investigations carried out, to date, on direct shear specimens include the works of Van de Loock (1987), Valle and Büyüköztürk (1993), Balaguru and Dipsia (1993), Khaloo and Kim (1997) and Mirsayah and Banthia (2002). In these studies various steel fibre volumes were randomly added to the matrix. Most of
the researchers conducted experimental work which was used in the derivation of their proposed model. The work of Balaguru and Dipsia was concerned solely with an experimental investigation of the behaviour of steel fibre reinforced high strength semi-lightweight concrete and did not propose a model.

Van de Loock (1987) considered the combined actions of shear and normal force in pre-cracked push-off specimens including hooked-ended steel fibres with a diameter of 0.8 mm and total length of 60 mm. The concrete compressive cube strength was 35 MPa and the equivalent cylinder strength is 44 MPa. The different fibre volume concentrations, $\rho_f$, considered in the tests were 0.0, 0.005 and 0.010. Van de Loock used a regression analysis to obtain the following equations for concrete shear and normal strengths without fibres:

\[
\tau = -1.21 + \left(3.83 w_w^{-1.26} - 1\right)w_v \\
\sigma_n = -1.82 + \left(3.27 w_w^{-1.03} - 1\right)w_v
\]

(Eq. 1.1) (Eq. 1.2)

where $\tau$ is the shear stress, $w_w$ is the crack width, $w_v$ is the vertical shear displacement and $\sigma_n$ is the normal stress. It was found that the influence of steel fibres on normal stress was small allowing the relationship between shear stress and displacements for steel fibre reinforced concrete to be expressed as:

\[
\tau = -1.21 + \left([4 + 128 \rho_f]w_w^{-1.18} - 1\right)w_v
\]

(Eq. 1.3)

Van de Loock stated that Eq. 1.3 underestimated the shear stress when compared to the experimental results. However, the author does not give an indication of the extent of this underestimation. The application for each of the proposed equations was restricted to the parameters of the test, viz, a matrix with 35 MPa compressive cube strength, initial crack width of 0.02 mm, the aggregates used in the testing programme and the range of crack openings and shear displacements tested. The restrictive nature of the Van de Loock equation makes them unsuitable in the development of a general model other than for comparison with the particular case.

Valle and Bilyükoztürk (1993) looked into the behaviour of steel fibres as well as polypropylene fibres subjected to direct shear in both normal strength ($f_{cm} = 29$ MPa) and high-strength ($f_{cm} = 80$ MPa) concrete. The effects from the addition and absence of stirrups across the shear plane were also studied. Crimped-ended steel fibres with a length of 30.5 mm and diameter 0.5 mm were used in the tests with a fibre volume
fraction of $\rho_f = 0.01$. Valle and Büyükoztürk used a shear transfer model to predict the stress-strain behaviour of plain and fibre reinforced concrete. The model was based on truss theory and took into account the softening component of concrete in compression due to the propagation and interaction of cracks. The method is comprehensive and involves a computation routine which requires 11 non-linear equations to be solved. Even though good agreement was obtained between the experimental results and the proposed model, the Valle and Büyükoztürk technique cannot be readily applied and the results obtained are neither easily identifiable nor convenient in a shear stress-strain format.

Khaloo and Kim (1997) proposed an empirical shear transfer model to predict the shear stress and shear displacement behaviour. The model is divided into two linear stages, up to cracking and failure at ultimate shear stress. Good agreement was obtained between the experimental results and the model. However, it was concluded in the report by Lee and Foster (2006a) that the Khaloo and Kim tests are of limited research value due to the mode of failure and the limited shear displacements measured.

Mirsayah and Banthia (2002) performed shear tests on beam specimens in preference to the double L-shaped specimens due to concerns that push-off specimens did not accurately replicate pure shear once cracking had taken place. In the Mirsayah and Banthia model, regression techniques were used to predict the ultimate shear stress of a fibre reinforced matrix as a function of $\rho_f$. For flattened-ended fibres the ultimate stress expression was:

$$\tau_{\text{max}} = 7.5 + 423\rho_f \quad \text{(Eq. 1.4)}$$

and the ultimate stress expression for the crimped fibres used in the tests was:

$$\tau_{\text{max}} = 7.5 + 29\sqrt{\rho_f} \quad \text{(Eq. 1.5)}$$

The Mirsayah and Banthia equations provide good agreement with the experimental results. However, without comparative control tests between the beam approach and the double L-shaped specimens, no comparisons are possible across studies, so the model cannot be evaluated against the double L-shaped specimens experimental results obtained in this study.

The application of the above models for general use is limited due to their restrictive or cumbersome nature and is not appropriate for use as a general design model. In
the following chapters a general design model is developed; named the Variable Engagement Model II. Throughout this study the terms mortar matrix or concrete matrix or matrix are taken to be synonymous.
CHAPTER 2

MODEL DEVELOPMENT

2.1 General

It is generally acknowledged that fibres function as a bridging mechanism across a crack, facilitating the transfer of stresses and imparting resistance to crack opening or sliding depending upon the mode of failure. The maximum bridging stress achievable depends on a number of aspects, such as, the quantity of fibres, fibre type, fibre material properties, fibre orientation with respect to the crack surface and pullout behaviour and fracture performance.

2.2 Model Assumptions

In the development of the Variable Engagement Model II outlined in this report, the following assumptions are made:

(i) the behaviour of a fibre reinforced composite may be obtained by a summation of its individual components, viz, the effects of the unreinforced matrix and each individual fibre can be added over the failure surface to reproduce the overall behaviour of the composite;

(ii) the geometric centres of the fibres are uniformly dispersed in space and all fibres have an equal probability of being oriented in any direction;

(iii) all fibres pullout from the side of the crack with the shorter embedded length while the longer side of the fibre remains rigidly embedded in the matrix;

(iv) displacements due to elastic strains taking place within the fibres are small in comparison to the displacements arising from movement occurring between the fibres and the matrix; and

(v) the energy expended by bending of fibres compared to that of pullout of the fibres is small and can be neglected.
2.3 Fibre Engagement Length

The shear fibre engagement length can be described as the vertical displacement of a fibre when it becomes effectively engaged in the shear carrying mechanism. Throughout the report the terms “shear engagement length” and “engagement length” can be considered synonymous. The crack sliding displacement at the point of fibre engagement is designated as \( w_{es} \). For the VEMII the fibre is taken to be effectively engaged in shear at a point corresponding to 50% of the peak load for the individual fibre. For the hooked-ended and straight fibres tested in this report the engagement relationship is shown in Figure 2.1 for mortar with a compressive strength of 43 MPa. In the figure the continuous function shown in solid outline is the fibre engagement function and is given by:

\[
\frac{w_{es}}{l_f} = \alpha_H \tan^2 \left( \frac{\pi}{4} - \frac{\theta}{2} \right)
\]

(Eq. 2.1)

where \( \alpha_H \) is the direct shear engagement constant, taken as 0.0075 for the discrete fibre tests conducted in this study, \( l_f \) is the fibre length and \( \theta \) is the fibre angle of orientation.

![Figure 2.1 Fibre engagement ratio at 50% peak load against angle of orientation.](image-url)

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orientation measured from a line drawn perpendicular to the loading plane expressed in radians; with a positive angle taken as an angle where the fibre has a component angle aligned with the direction of loading and a negative angle where the fibre has a component angle that opposes the direction of loading. A fibre at an angle of zero degrees is normal to the loading direction. It can be seen that Eq. 2.1 satisfies the required boundary conditions, viz:

(i) \[ \theta \to -\pi/2 \quad \frac{w_s}{l_f} \to \infty; \text{ and} \]
(ii) \[ \theta \to +\pi/2 \quad \frac{w_s}{l_f} \to 0. \]

### 2.4 Fibre Engagement Angle

In a randomly oriented fibre composite material subjected to direct shear, fibres oriented greater than a critical angle, \( \theta_{crit} \), are engaged and may carry load; whereas fibres at angles less than \( \theta_{crit} \) are yet to be engaged. The angle at which fibres become engaged, that is, when the fibres become active and contribute to the strength of the matrix engagement is obtained by substituting \( \theta = \theta_{crit} \) in Eq. 2.1 and is, thus, given by:

\[
\theta_{crit} = \frac{\pi}{2} - 2 \tan^{-1} \left( \frac{w_s}{\alpha_l l_f} \right)
\]

(Eq. 2.2)

where \( w_s \) is the crack sliding displacement (CSD). For a given CSD, as \( \alpha_l \) decreases \( \theta_{crit} \) decreases. Thus, \( \alpha_l \) is a material measure of the resistance to slip between the fibre and matrix.

The limiting angle, \( \theta_{lim} \) is the maximum angle at which the fibres can be engaged in direct shear. For fibres with \( \theta \leq \theta_{lim} \) engagement is not possible whereas fibres with \( \theta > \theta_{lim} \) can become engaged or may have pulled out from the matrix, depending on the initial embedment length of the fibre.

### 2.5 Model at Peak Load

In the report by Lee and Foster (2006a) the failure modes for the discrete fibre specimens were noted in Tables 3.5 to 3.8 while in the report by Lee and Foster (2006b) the failure modes for the non-destructive specimens were given in Table 3.2.
The modes consisted of pullout, fracture and a combination of pullout and fracture. At the peak load, the majority of fibres remain intact and the mode of failure can be predicted by the shape of the load per fibre against vertical displacement curve. Fibres which exhibit a distinct snubbing zone are more likely to fail by fracture. The non-destructive investigation can be used to assist with the development of the VEMII predictions at peak load.

From the detailed fibre arrangements shown in Appendix A (Figures A.1 to A.8) it is possible to divide a fibre into a number of distinct zones which represent the types of fibre performance present. The different zones, illustrated in Figure 2.2, are the hook, the straight, the snubbing and the bridging zones. The hook zone corresponds to the hook present at the ends of a hooked-ended fibre and, thus, for a straight fibre the length of the hook zone is zero. The straight segment is the original straight, unbent, section of a fibre. The snubbing portion is the bent part of the fibre still positioned within the matrix. The bridging segment is the exposed section of fibre which spans across the shear plane linking the two halves of the specimen and can be composed of a combination of bent and "straight" portions of fibre. It is not possible to obtain an image of the fibre bridging parts from the gamma ray imaging process as the film exposure time is calculated for the gamma rays to pass through the denser matrix. The profile of the exposed fibres in these regions has been carefully assessed to reflect the visual observations between the two shear planes.

Each of the zones can be assigned their own shear contribution as shown in Figure 2.3. The shear stresses are assumed to be the average uniform value acting on the fibre perimeter along the length of the fibre component concerned. In the subsequent figures the following notations are used:

\[ a_{snub} = \text{proportion of snubbing zone engaged} \]
\[ l_{br} = \text{total length of bridging zone} \]
\[ l_{hook} = \text{total length of hook} \]
\[ l_{snub} = \text{total length of snubbing zone} \]
\[ l_{st} = \text{total length of straight zone} \]
\[ \tau_{hook} = \text{shear stress along the embedded hook fibre end under direct shear} \]
\[ \tau_{snub} = \text{shear stress along the fibre length in the snubbing zone under direct shear} \]
\[ \tau_{st} = \text{shear stress along the embedded straight fibre lengths under direct shear} \]
Figure 2.2  Fibre performance zones.
Figure 2.3 Fibre shear contribution associated with each performance zone for an end hooked fibre.

A part of the fibre perimeter in the snubbing zone is not in contact with the matrix. The shearing action at the face of the shear plane forces the mortar in its path to be progressively removed leaving a partial perimeter of the fibre in contact with the matrix (Refer Figure 2.2 Detail 1 and Section 1). It is not possible to ascertain the extent of contact surface with any degree of accuracy since the removal of mortar in the snubbing zone does not take place uniformly. More than likely the portion of fibre closest to the shear plane would have the least contact perimeter whereas the part of fibre at the start of the snubbing zone would be the greatest. In the calculations which follow an average value is taken for the fibre shear stress in the snubbing zone and is applied to the entire fibre perimeter within the snubbing zone.

In Lee and Foster (2006a) it was noted that the embedment length of the fibre each side of the shear plane did not significantly influence the peak load. The contribution of the fibre perimeter stresses, $\tau_{st}$, in the straight zones is, thus, considered small and is ignored in the development of the VEMII model.

The experimental peak loads ($P_{f0}$) against angle of orientation from Lee and Foster (2006a) is reproduced in Figure 2.4(a) and includes the peak loads for the gamma ray specimens. It is seen that for pullout failure, the peak loads in the fibres, $P_{f0}$, can be approximated by a linear function in $\theta$ and that the difference between the straight
Figure 2.4  Plot of peak load per fibre against angle of orientation; (a) experimental data with outlines for hooked-ended and straight fibre plots indicated; (b) simplified model arrangements.
and hooked-ended fibres is a constant 360 N. This represents the effect of the hook. In the pullout region the plots in Figure 2.4 can be expressed as:

\[
(P_f \theta)_{straight} = 103 \left( \frac{\pi}{2} - \theta \right) \text{ N for } \theta \geq -\frac{11\pi}{36} \quad \text{(Eq. 2.3)}
\]

and

\[
(P_f \theta)_{hooked} = (P_f \theta)_{straight} + 360 \text{ N for } \theta \geq -\frac{\pi}{6} \quad \text{(Eq. 2.4)}
\]

In the simplified model arrangements for the fibres, shown in Figure 2.4(b), the strength of the fractured fibres is ignored. This assumption is deemed reasonable in the context of the model development as once fractured the fibre no longer contributes to the strength of the integrated matrix. It can be construed from the figure that there are four possible stages that a discrete fibre can exist in the VEMII. They are:

(i) the fibre is not engaged and may enhance the strength of the matrix when engagement is activated;
(ii) the fibre fractured upon engagement and therefore does not contribute to the strength of the matrix;
(iii) the fibre is engaged and contributing to the strength of the matrix; and
(iv) the fibre was engaged but has since pulled out from the matrix and no longer adds to the strength of the matrix.

The significance of stage (ii) is that data or any curve fits in the fibre fracture zone (Figure 2.4) do not need to be considered in the model development as the fibre contribution in this region is taken as zero. In practice the fibres which are located in the fracture region will contribute to the strength of the matrix. However the contribution is minimal over the small range of displacements from engagement to fracture and is therefore inconsequential to the overall response.

The peak load \((P_f 0)\) for the fibres is the summation of the end hook contribution \((P_{hook})\) and the snubbing \((P_{snub})\) contribution, expressed as follows:

\[
P_f 0 = P_{hook} + P_{snub} \quad \text{(Eq. 2.5)}
\]

where \(P_{hook}\) is zero for a straight fibre.
2.5.1 Hook Contribution

The hook shear stress, $\tau_{\text{hook}}$, is the difference between the peak load for the hooked-ended and straight fibre specimens divided by the fibre circumference ($\pi d_f$) and hook length (6 mm). The relationship can be expressed as:

$$\tau_{\text{hook}} = \frac{P_{\text{hook}}}{\pi d_f l_{\text{hook}}} \quad \text{(Eq. 2.6a)}$$

For the fibres used in this study the contribution of the end hook is approximately constant with $P_{\text{hook}} = 360$ N (Figure 2.4a and Eq. 2.4). The resulting hook shear stress is calculated as:

$$\tau_{\text{hook}} = \frac{360}{\pi (0.9)^2 (6)} = 21.2 \text{ MPa} \quad \text{(Eq. 2.6b)}$$

Taking the mean tensile strength of the mortar as $f_{\text{ct}} = 0.5 \sqrt{f_{\text{cm}}}$, (AS 3600-2005 Draft) where $f_{\text{cm}}$ is the mean concrete compressive strength, gives:

$$f_{\text{ct}} = 0.5 \sqrt{43} = 3.3 \text{ MPa} \quad \text{(Eq. 2.7)}$$

The hook stress expressed as a multiple of the tensile strength of the mortar is:

$$\tau_{\text{hook}} = 6 f_{\text{ct}} \text{ MPa} \quad \text{(Eq. 2.8)}$$

Fibre fracture ($P_{\text{frac}}$) will take place when the tensile strength in the fibre exceeds its fracture strength given by $\sigma_{f_i}$. For the fibres used in this study:

$$P_{\text{frac}} = \sigma_{f_i} A_f$$

$$= 1210 \frac{\pi (0.9)^2}{4} \text{ N}$$

$$= 770 \text{ N} \quad \text{(Eq. 2.9)}$$

Comparing the above fracture value with the hook contribution it can be seen that without the snubbing contribution the hook would pullout from the matrix prior to fracture.
2.5.2 Snubbing Contribution

The effects from snubbing are the principal mode of behaviour for a straight fibre. There are two possible simplified modelling options that could be considered in the determination of the snubbing shear stress:

(i) a constant value for \( l_{snub} \) with \( \tau_{snub} \) taken as a function of \( \theta \); or

(ii) a constant value for \( \tau_{snub} \) with \( l_{snub} \) taken as a function of \( \theta \).

The preferred approach is the first option since it has the advantage of maintaining a constant snubbing length over which the bond stresses apply and is, thus, not a function of the fibre orientation.

The snubbing zone lengths for the fibres close to peak load are shown in Appendix A for the gamma ray specimens (Lee and Foster, 2006b) and plotted in Figure 2.5. The average snubbing zone length, \( l_{snub} \), for the range \(-\pi/2 \leq \theta \leq +\pi/2\) is approximately \( 3d_f = 2.7 \text{ mm} \). The snubbing shear stress is expressed as:

\[
\tau_{snub} = \frac{P_{snub}}{\pi d_f l_{snub}}
= \frac{103 \left( \frac{\pi}{2} - \theta \right)}{\pi d_f (3d_f)}
= \frac{103 \left( \frac{\pi}{2} - \theta \right)}{3 \pi d_f^2}
= \frac{103 \left( \frac{\pi}{2} - \theta \right)}{3 \pi (0.9)^2} \text{ MPa}
= 13.5 \left( \frac{\pi}{2} - \theta \right) \text{ MPa} \quad \text{(Eq. 2.10)}
\]

The snubbing shear stress expressed as a multiple of the tensile strength of the mortar is:

\[
\tau_{snub} = 4 f_{ct} \left( \frac{\pi}{2} - \theta \right) \text{ MPa} \quad \text{(Eq. 2.11)}
\]
Figure 2.5  Plot of snubbing length against orientation angle.

In Figures A.1 to A.8 it is evident that fibres subjected to snubbing undergo significant bending \((M_{snub})\) in addition to axial tension \((P_{snub})\). The combined actions can be expressed as:

\[
\frac{P_{snub}/A_f}{\sigma_{fu}} + \frac{M_{snub}/Z_f}{\sigma_{fu}} \leq 1 
\]

(Eq. 2.12)

where \(Z_f\) is the fibre section modulus. Fibre fracture will take place when the combined actions exceed unity in Eq. 2.12. The axial component in Eq. 2.12 can be rewritten as:

\[
c = \frac{\sigma_{fu}(\theta)}{\sigma_{fu0}}
\]

(Eq. 2.13)

where \(c\) is the axial fracture ratio, \(\sigma_{fu}(\theta)\) is the fracture strength for a fibre oriented at an angle \(\theta\) to the shear plane and \(\sigma_{fu0}\) is the fibre strength for zero bending stress.
For fracture of the straight fibre in Figure 2.4 the value of $c$ is:

$$
c_{\theta=-55^0} = \frac{261}{\left(\frac{\pi (0.9)^2}{4}\right)} \frac{1210}{1210} = 0.34
$$

(Eq. 2.14)

The above calculation demonstrates the importance of including bending effects in any fibre model that includes fibre fracture.

In Figure 2.6a and 2.6b, the discrete fibre arrangement during the uncracked phase and at peak load are shown. The fibre model layout at the snubbing region is illustrated in Figure 2.6c where $R$ is the radius of fibre bend, $\alpha_{snub}$ is the enclosed angle of the fibre bend and $l_{sv}$ is the vertical snubbing length. From the figure the following can be derived:

$$l_{sv} = R \cos\left(\frac{\theta}{\pi}\right)
$$

(Eq. 2.15)

and

$$l_{snub} = R \alpha_{snub}
$$

$$= R \left(\frac{\pi}{2} - \theta\right)
$$

(Eq. 2.16a)

so,

$$R = \frac{l_{snub}}{\left(\frac{\pi}{2} - \theta\right)}
$$

(Eq. 2.16b)

Substituting for $R$ in Eq. 2.15 the vertical length of the snubbing zone is evaluated as:

$$l_{sv} = \frac{l_{snub} \cos\left(\frac{\theta}{\pi}\right)}{\left(\frac{\pi}{2} - \theta\right)}
$$

(Eq. 2.17)

The curvature $\kappa$ is the inverse of the radius of curvature:

$$\kappa = \frac{\left(\frac{\pi}{2} - \theta\right)}{l_{snub}}
$$

(Eq. 2.18)
Figure 2.6  Discrete fibre arrangements for: (a) uncracked; (b) bending layout at peak; (c) model configuration for fibre in snubbing region for one side of the shear plane.
However to account for the difference between elastic theory and the plastic deformation of the fibre a \( \frac{1}{\beta} \) term is included in the above equation to give:

\[
\kappa = \frac{1}{\beta} \left( \frac{\pi}{2} - \theta \right) \tag{Eq. 2.19}
\]

where \( \beta \) is a fibre fracture bending coefficient.

Assuming the portion of the fibre over which bending occurs has a constant moment over the radius of curvature, the curvature can be expressed as the relationship between bending strain (\( \varepsilon_b \)) and fibre diameter as follows:

\[
\kappa = \frac{\varepsilon_b}{\left( \frac{d_f}{2} \right)} \tag{Eq. 2.20}
\]

Substituting Eq. 2.20 into Eq. 2.19 gives:

\[
\varepsilon_b = \frac{1}{\beta} \frac{d_f \left( \frac{\pi}{2} - \theta \right)}{2 l_{snub}} \tag{Eq. 2.21}
\]

For the consideration of the bending effects a number of different behavioural models will be investigated. They are:

(i) the elastic-brittle model;
(ii) the rigid-plastic model; and
(iii) the elastic-plastic model.

For an elastic-brittle material once the yield strain (\( \varepsilon_{fy} \)) has been reached the fibre fractures (See Appendix B, Figure B.1). In the rigid-plastic model the stress in the fibre is taken as equal to the yield stress for any strain other than zero (See Appendix C, Figure C.1). For a zero strain the stress is undefined. For an elastic-plastic material, the stress-strain is linear up to the yield point and constant thereafter (See Appendix D, Figure D.1). The results for each of the behavioural models are given in the following sections.


2.5.3 Interaction of Fibre Fracture with Bending

Elastic-brittle Model

Elastic-brittle material behaviour is a lower bound approach which means that without adjustment the true solution lies above the results predicted by this method. The investigation into the method is outlined in detail in Appendix B. In the appendix the bending moment factor term $\bar{d}$ is the ratio between the position of the neutral axis and half the fibre diameter. It is an indicator of the position of the neutral axis and can be used to determine the level of bending moment present in the fibre. For the model the relationship between axial tension ratio ($c$) and $\bar{d}$ is shown in Figure B.11 and against strain ratio in Figure B.12.

The axial fracture ratio for this approach is 0.38 ($\sigma_{f0} = \sigma_{y} = 1085$ MPa) and from Appendix B the bending moment factor is:

$$
\bar{d} = \frac{a_e \varepsilon_{fy}}{\varepsilon_{be}} - 1 
$$

(Eq. 2.22)

The value for the unknown $\beta$ term can be obtained from Figure B.12 as follows:

$$
\beta = \frac{d_f \left( \frac{\pi}{2} - \theta \right)}{2 l_{sub}}
\begin{align*}
&= \frac{1}{0.62(0.0069)} \left( \frac{0.9 \left( \frac{\pi}{2} - \left( -\frac{\pi}{4} \right) \right)}{2 (2.7)} \right) \\
&= 92
\end{align*}
$$

(Eq. 2.23)

The resulting relationship between the axial tension ratio and orientation angle is shown in Figure 2.7. The interaction diagram for the straight fibres is shown in Figure B.15.
Figure 2.7  Plot of axial tension ratio against orientation angle for straight fibres (elastic-brittle model).

Rigid-plastic Model

The model based on rigid-plastic behaviour is an upper bound approach which, without adjustment, implies that the true solution lies below the results predicted by this technique. The detailed investigation into the method appears in Appendix C. For the model the relationship between the axial tension ratio, $c$ (Eq. 2.14) and $\bar{d}$ is shown in Figure C.6 and against strain ratio in Figure C.7.

Adopting the same procedure as for the elastic-brittle approach, the value for the unknown $\beta$ term can be obtained from Figure C.7 as follows:

$$\beta = \frac{1}{\varepsilon_{bp}} \frac{d_f \left( \frac{\pi}{2} - \theta \right)}{2l_{snub}}$$

$$= \frac{1}{0.787 \varepsilon_{fu}} \frac{d_f \left( \frac{\pi}{2} - \theta \right)}{2l_{snub}}$$  \hspace{1cm} (Eq. 2.24a)
\[
\beta = \frac{1}{0.787(0.0244)} \frac{0.9 \left( \frac{\pi}{2} - \left( -\frac{\pi}{4} \right) \right)}{2 \left( 2.7 \right)} = 20
\]

(Eq. 2.24b)

The resulting relationship between the axial tension ratio and orientation angle is shown in Figure 2.8. The interaction diagram for the straight fibres is shown in Figure C.10.

![Figure 2.8](image)

**Figure 2.8** Plot of axial tension ratio against orientation angle for straight fibres (rigid-plastic model).

**Elastic-plastic Model**

For an elastic-plastic material, once the yield strain \( (\varepsilon_y) \) has been exceeded the stress distribution is no longer linear but remains at its yield value during the ductile phase until the fracture strain, \( \varepsilon_{fu} \), is reached. The detailed investigation into the method appears in Appendix D. For the model the relationship between axial tension ratio \( (c = 0.34) \) and \( d \) is shown in Figure D.12 and against strain ratio in Figure D.13.
The value for the unknown $\beta$ term can be obtained from Figure D.13 as follows:

$$\beta = \frac{1}{\epsilon_{bep}} \frac{d_f \left( \frac{\pi}{2} - \theta \right)}{2 l_{stub}}$$

$$= \frac{1}{0.78 \epsilon_{fu}} \frac{d_f \left( \frac{\pi}{2} - \theta \right)}{2 l_{stub}}$$

$$= \frac{1}{0.78 (0.0244)} \frac{0.9 \left( \frac{\pi}{2} - \left( -\frac{\pi}{4} \right) \right)}{2 (2.7)}$$

$$= 21$$  \hspace{1cm} (Eq. 2.25)

The resulting relationship between the axial tension ratio and orientation angle is shown in Figure 2.9. The interaction diagram for the straight fibres is shown in Figure D.16.

A comparison of each of the models reveals that the elastic-brittle model covers the range of orientation angles between $+0.30^\circ \leq \theta \leq +\pi/2$ (Figure 2.7). The rigid-plastic model is applicable to angles in the range $-1.42^\circ \leq \theta \leq +\pi/2$ (Figure 2.8) and the elastic-plastic model is restricted to orientation angles in the range $-1.44^\circ \leq \theta \leq +\pi/2$ (Figure 2.9). Outside the lower limit, the fibres must fracture, due to the effect of snubbing, irrespective of the contribution of any hook portion.

The axial tension ratio curves and the interaction curves for the different models are compared in Figure 2.10. It is seen that the elastic-brittle model is too conservative for determining whether, or not, fibre fracture affects the shear stress versus pullout behaviour. For an axial tension ratio of 0.5, the elastic-plastic model moment ratio is approximately 46% higher and the rigid-plastic model moment ratio is about 53% higher than the moment ratio for the elastic-brittle model.
Figure 2.9  Plot of axial tension ratio against orientation angle for straight fibres (elastic-plastic model): (a) bending moment factors; and (b) range of applicable orientation angles.
Figure 2.10  Comparison of plots for elastic-brittle, elastic-plastic and rigid-plastic models: (a) axial tension ratio curves; (b) interaction curves.
2.5.4 Critical Fibre Length for Fibre Fracture

In the formulations that follow, it is assumed that the effect of fibre fracture does not influence the outcomes of the model. This is the case provided that the length of the fibre is such that fibres aligned at high (in the absolute sense) negative angles pullout before engagement or fracture. Based on the assumption that fibres pullout from the shorter embedded side, then a fibre will not fracture at the point of engagement if:

\[
\tau_b \pi d_f l_a \leq \frac{\sigma_{fu}}{4} \pi \frac{d_f^2}{4} \\
l_a \leq \frac{d_f}{4} \frac{\sigma_{fu}}{\tau_b}
\]  

(Eq. 2.26)

where \( l_a \) is the fibre embedment length (measured on the short side), \( \sigma_{fu} \) is the effective ultimate tensile strength of the fibre and \( \tau_b \) is the average bond stress along the fibre length. As the longest embedment length (on the short side) is \( l_f / 2 \), we can determine from Eq. 2.26 whether, or not, fibre fracture will influence the resulting \( \tau \)-\( w_s \) curve.

For the values of \( \sigma_{fu} \) in Eq. 2.26, the angle of the fibre to the cracking plane affects the effective axial strength of the fibre when bending effects are considered and various models encompassing this effect were presented in the previous section. The models, however, are complex and before adding this additional level of complexity, the impact of fibre fracture and bending on the resulting \( \tau \)-\( w_s \) curve should first be assessed.

Looking first at the fibre fracture conditions, it is observed that fibre fracture is more likely for the more negative angled fibres and these fibres do not engage until large sliding displacements first occur. Fibres at the large angles will pullout before engagement and do not contribute to the strength.

Taking the effective fibre strength with bending as \( \sigma_{fut} = 0.5\sigma_{fu} \), where \( \sigma_{fu} \) is the ultimate tensile strength of the fibre and substituting the maximum fibre embedment length \( l_{a,\text{max}} = l_f / 2 \) into Eq. 2.26 gives:

\[
l_f < l_e = \frac{d_f}{4} \frac{\sigma_{fu}}{\tau_b}
\]  

(Eq. 2.27)
where $l_c$ is the critical fibre length. The value of $\sigma_{f_{iu}} = 0.5 \sigma_{f_{iu}}$ equates to a fibre angle of approximately $-0.56$ radians ($-32^\circ$) in the elastic-plastic fibre fracture model for the single fibre tests in this study (see Figure 2.9). If the inequality of Eq. 2.27 is violated then a significant portion of the fibres may fracture and fibre fracture should be considered in calculating the $\tau_w$ curve.

### 2.5.5 Fibre Contribution Excluding Fibre Fracture

The discrete fibre contribution for VEMII can be calculated as the summation of the hook and snubbing contributions for each fibre as follows:

$$P_f = k \pi d_f \left( l_{\text{hook}} \tau_{\text{hook}} + l_{\text{snub}} \tau_{\text{snub}}(\theta) \right)$$  
(Eq. 2.28)

where $k$ is a fibre engagement factor equal to one when the fibre is engaged and the length of the embedded fibre is such that $l_a \geq l_{\text{hook}} + l_{\text{snub}}$ and equal to zero for a fibre yet to be engaged ($w_s < w_{es}$) or has already pulled out from the matrix (that is, $w_s \geq l_a$). For straight fibres $l_{\text{hook}} = 0$ in Eq. 2.28.

When the shear plane is close to the end of a fibre an overlap of the snubbing and hook performance zones takes place ($l_a < l_{\text{hook}} + l_{\text{snub}}$). The different arrangements are shown diagrammatically in Figure 2.11. The shear load under these circumstances can be evaluated as:

$$\bar{P}_f = P_f \left( \frac{l_a}{l_{\text{hook}} + l_{\text{snub}}} \right)$$  
(Eq. 2.29)

where is the $\bar{P}_f$ is the pullout force due to overlapping snubbing and hook performance zones.
Figure 2.11 Peak loads per fibre for shear plane near fibre end: (a) zero length for shorter straight zone; (b) reduced snubbing length; (c) no snubbing zone; (d) reduced hook length.

2.6 Number of Fibres Crossing the Shear Plane

For the individual fibre contribution to be integrated across a crack plane, the number of fibres in a three-dimensional matrix crossing the cracking plane first needs to be calculated. Aveston and Kelly (1973) showed that the number of aligned fibres for a one-dimensional fibre distribution, $N_{at}$, crossing a crack of unit area is obtained by calculating the total mass of fibres in a matrix over a unit area and dividing by the mass of a single fibre. The resulting equation gives the number of aligned fibres for a unit area as:

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\[ N_{al} = \frac{\rho_f}{A_f} \]  \hspace{1cm} \text{(Eq. 2.30)}

where \( \rho_f \) is the volume fraction of fibres and \( A_f \) is the cross-sectional area of a single fibre.

For the case of fibres crossing a plane of unit area at angle \( \phi \) (Figure 2.12), the number of fibres cutting the plane is \( \rho_f \sin \phi / A_f \). Now considering a small change of angle \( d\phi \), the change in the number of fibres cutting the plane is:

\[ \Delta N = \frac{\rho_f}{A_f} \sin \phi \frac{d\phi}{(\pi/2)} \]  \hspace{1cm} \text{(Eq. 2.31)}

For fibres randomly orientated in two-dimensions, the total number of fibres crossing a plane of unit area is:

\[ N = \int_0^{\pi/2} \Delta N \, d\phi = \frac{2 \rho_f}{\pi A_f} \int_0^{\pi/2} \sin \phi \, d\phi \]  \hspace{1cm} \text{(Eq. 2.32a)}

giving

\[ N = \frac{2 \rho_f}{\pi A_f} \]  \hspace{1cm} \text{(Eq. 2.32b)}

Figure 2.12  Plan view of fibres crossing a plane of unit area (Aveston and Kelly, 1973).
A similar derivation can be undertaken for randomly orientated fibres in three-dimensional space (Figure 2.13) where, for a plane of unit area:

$$N = \frac{\rho_f}{A_f} \int_0^\pi \cos \phi \sin \phi \, d\phi = \frac{\rho_f}{2 A_f}$$  \hspace{1cm} \text{(Eq. 2.33)}$$

For a shear plane with a cross-sectional area of $A_c$, the total number of randomly distributed fibres in three-dimensional space crossing the plane is:

$$N = \frac{\rho_f A_c}{2 A_f}$$  \hspace{1cm} \text{(Eq. 2.34)}$$

The calculated total number of fibres crossing the shear plane for the randomly distributed fibre specimens tested for the series of experiments described in Lee and Foster (2006a) is given in Table 2.1.

Figure 2.13  View of fibres crossing a crack in a three-dimensional random composite (Aveston and Kelly, 1973).
Table 2.1  Calculated total number of fibres crossing the shear plane for randomly distributed fibre specimens.

<table>
<thead>
<tr>
<th>Specimen</th>
<th>$\rho_f$</th>
<th>Average Shear Plane Area, $A_c$ (mm$^2$)</th>
<th>Number of fibres crossing the shear plane, $N$ (Eq. 2.34)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Hooked ends</td>
</tr>
<tr>
<td>DSR-H-0.5</td>
<td>0.005</td>
<td>6,525</td>
<td>1</td>
</tr>
<tr>
<td>DSR-H-1.0</td>
<td>0.010</td>
<td>6,500</td>
<td>69</td>
</tr>
<tr>
<td>DSR-H-1.5</td>
<td>0.015</td>
<td>6,500</td>
<td>137</td>
</tr>
<tr>
<td>DSR-H-2.0</td>
<td>0.020</td>
<td>7,000</td>
<td>205</td>
</tr>
<tr>
<td>DSR-S-0.5</td>
<td>0.005</td>
<td>6,800</td>
<td>295</td>
</tr>
<tr>
<td>DSR-S-1.0</td>
<td>0.010</td>
<td>6,000</td>
<td>541</td>
</tr>
<tr>
<td>DSR-S-1.5</td>
<td>0.015</td>
<td>7,300</td>
<td>955</td>
</tr>
<tr>
<td>DSR-S-2.0</td>
<td>0.020</td>
<td>6,250</td>
<td>1,742</td>
</tr>
</tbody>
</table>

2.7  Shear Stress for Randomly Distributed Fibres (VEMII)

2.7.1 General

It is hypothesized that the peak load for a discrete fibre given in Section 2.5.5 can be used to determine the shear stress $\tau$ (over a plane of unit area) by summing the fibre ($\tau_f$) and matrix ($\tau_c$) contributions:

$$\tau = \tau_f + \tau_c$$  (Eq. 2.35)
For integration of the fibre contribution two different approaches are considered (Figure 2.14):

(i) a lumped bond stress approach; and
(ii) a uniform bond stress approach.

The lumped approach assumes that the bond is concentrated at the hook and snubbing zones, as observed in the discrete fibre tests. The uniform bond approach assumes that the bond is smeared uniformly along the length of the fibre.

The behaviour of the fibre matrix composite can be described by the shear stress versus crack sliding displacement (CSD) shown in Figure 2.15. The crack sliding displacement, \( w_s \), is the displacement which occurs beyond the peak shear stress and excludes the effects from any bulk matrix strain on the displacement. The crack sliding displacement, calculated as shown in Figure 2.16, is:

\[
w_s = w_v - w_{v\text{ peak}} + \frac{w_{\text{el}}}{\tau_1} \left( \tau_{\text{peak}} - \tau_v \right)
\]

(Eq. 2.36)

where \( w_v \) is the measured displacement, \( w_{v\text{ peak}} \) is the measured displacement corresponding to the peak load, \( w_{\text{el}} \) is a point on the elastic curve corresponding to the point \( \tau_1 \) (see Figure 2.16) and \( \tau_{\text{peak}} \) is the maximum shear stress.

![Figure 2.14 Hooked-ended fibre shear contribution: (a) general arrangement; (b) lumped bond stress approach; and (c) uniform bond stress approach.](image)

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Figure 2.15  Plot of shear stress against crack sliding displacement for the matrix, fibres and fibre matrix composite.

Figure 2.16  Arrangement of basic terms for the determination of crack sliding displacement.
2.7.2 Mortar Component

In Figure 2.17 the shear stress versus vertical displacement is plotted for an unreinforced mortar specimen tested in this study. The dimensions of the specimen were as for the discrete fibre specimens (see Figure 3.1, Lee and Foster, 2006a). In Figure 2.18 the shear stress ($\tau_c$) is plotted against the crack opening displacement ($w_s$), that is with the elastic component of the bulk material subtracted from the measured vertical displacement ($\delta_v$).

In the analyses that follow, the mortar contribution to the composite is taken as:

$$\tau_c = c_1 \tau_0 e^{-c_2 w_s}$$  \hspace{1cm} (Eq. 2.37)

where $\tau_0$ is the strength of the mortar with $\rho_f = 0.0$ and $c_1$ and $c_2$ are coefficients. The coefficient $c_1$ accounts for the beneficial effect of the fibres on the peak matrix strength. The coefficient $c_2 = 3.2$ is obtained by fitting the test results for $\rho_f = 0.0$ (see Figure 2.18). The peak matrix shear strength versus fibre reinforcement ratio is plotted in Figure 2.19. The figure gives:

$$\tau_0 = 4.4 \text{ MPa}$$  \hspace{1cm} (Eq. 2.38a)

$$c_1 = 1 + 72 \rho_f$$  \hspace{1cm} (Eq. 2.38b)

Figure 2.17 Average shear stress against vertical displacement for plain mortar.
Figure 2.18  Shear stress against crack sliding displacement for plain mortar.

Figure 2.19  Peak matrix shear stress against $\rho_f$. 

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2.7.3 Lumped Bond Shear Stress Model

Incorporating the number of fibres in the matrix from the work of Aveston and Kelly (1973) (Eq. 2.34) and integrating Eq. 2.28 over a plane of unit area, the shear stress contribution from the fibre component for the lumped bond model can be expressed as:

\[
\tau_f = \pi d_f \left[ l_{\text{hook}} \tau_{\text{hook}} + \frac{l_{\text{sub}}}{2} \tau_{\text{sub.ave}} \right] K_{\tau} \frac{\rho_f}{2A_f}
\]

\[
= \frac{2}{d_f} K_{\tau} \rho_f \left[ l_{\text{hook}} \tau_{\text{hook}} + \frac{l_{\text{sub}}}{2} \tau_{\text{sub.ave}} \right]
\]  

(Eq. 2.39)

where \(\tau_{\text{sub.ave}}\) is the average snubbing shear stress for all engaged fibres and \(K_{\tau}\) is the global orientation factor. The average snubbing shear stress is calculated as:

\[
\tau_{\text{sub.ave}} = 0.5 \tau_{\text{sub}} \left( \theta_{\text{critl}} \right)
\]  

(Eq. 2.40)

For the 60 mm fibres used in the discrete tests, substituting for \(\tau_{\text{sub}}\) (Eq. 2.10) and \(\theta_{\text{critl}}\) (Eq. 2.2) into Eq. 2.40, the average snubbing shear stress becomes:

\[
\tau_{\text{sub.ave}} = 13.5 \tan^{-1} \left( \frac{w_s}{\alpha_0 l_f} \right) \text{ MPa}
\]  

(Eq. 2.41)

and, thus, is a function of the current crack sliding displacement.

The global orientation factor can be determined using the rules of probability and is affected by the shape of the domain over which the orientation is considered. Using the fibre engagement model described by Eq. 2.1:

\[
K_{\tau} = \frac{1}{N} \sum_{i=1}^{N} k_i(w)
\]  

(Eq. 2.42)
where \( N \) is the number of fibres crossing a plane of unit area, \( k_i \) is the local orientation factor for the \( i \)th fibre and \( F \) is an end effect factor for fibres embedded at \( l_a < l_{hook} + l_{snub} \).

In the lumped model, \( k_i = 1 \) for all engaged fibres and \( k_i = 0 \) for fibres that have yet to be engaged or have pulled out from the matrix. The direct shear global orientation factor is then expressed as:

\[
K_r = F \frac{n}{N} \tag{Eq. 2.43}
\]

where

\[
\frac{n}{N} = \left( \frac{\pi - \theta_{crit}}{\frac{2}{\pi}} \right) p \tag{Eq. 2.44}
\]

and \( n \) is the number of fibres engaged and \( p \) is the proportion of fibres remaining embedded across a crack, for a given \( w_s \). With the assumption of a uniform distribution of fibres, \( p \) is:

\[
p = 1 - \frac{2w_s}{l_f} \tag{Eq. 2.45}
\]

Substituting Eqs. 2.2, 2.44 and 2.45 into Eq. 2.43, the global orientation factor is:

\[
K_r = F \frac{2}{\pi} \tan^{-1} \left( \frac{w_s}{\alpha_{II} l_f} \right) \left( 1 - \frac{2w_s}{l_f} \right) \tag{Eq. 2.46}
\]

An alternative formulation of the global orientation factor can be found from Eq. 2.2 and Foster et al. (2006):

\[
K_r = \lim_{{N \to \infty}} F \frac{1}{N} \left\{ \sum_{{\theta = -\pi/2}}^{\pi/2} k(w_s) \left( \frac{\text{d}\theta}{\Delta\theta} \right) \right\} \tag{Eq. 2.47a}
\]
\[
K_r = \lim_{N \to \infty} F \frac{1}{N} \left\{ \sum_{-\frac{\pi}{2}}^{\frac{\pi}{2}} k(w_s) \frac{\Delta \theta}{\pi} + \sum_{\theta_{critII}}^{\frac{\pi}{2}} k(w_s) \frac{\Delta \theta}{\pi} \right\}
\]

\[
= \lim_{N \to \infty} F \frac{1}{N} \left\{ 0 + \sum_{\theta_{critII}}^{\frac{\pi}{2}} k(w_s) \frac{\Delta \theta}{\pi} \right\} = \lim_{N \to \infty} F \frac{1}{N} \left\{ \frac{\pi}{\pi} \sum_{\theta_{critII}}^{\frac{\pi}{2}} k(w_s) \Delta \theta \right\}
\]

(Eq. 2.47b)

In the first line of Eq. 2.47b the first summation term between \(-\pi/2\) and \(\theta_{critII}\) is zero since it represents the circumstance for fibres that are yet to be engaged and, hence, not contributing to the strength of the matrix. Noting that \(N^{-1} \sum k(w_s) = k_{ave} \rho\) where \(k_{ave}\) is the average for all engaged fibres (\(k_{ave} = 1\) for the lumped approach) then Eq. 2.47b is rewritten as:

\[
K_r = F \rho \frac{1}{\pi} \sum_{\theta_{critII}}^{\frac{\pi}{2}} \Delta \theta = F \rho \left( \frac{1}{2} - \frac{\theta_{critII}}{\pi} \right)
\]

(Eq. 2.48)

which is the same as Eqs. 2.43 and 2.44.

The end effect factor can be determined with consideration of the ratio:

\[
R = \frac{l_{hook} + l_{snub}}{l_f/2 - w_s}
\]

(Eq. 2.49)

There are two cases requiring consideration:

(i) maximum fibre embedded length is greater than the sum of the hook and snubbing lengths; and
(ii) maximum fibre embedded length is less than the sum of the hook and snubbing lengths.

For these two conditions it can be determined that:

\[ F = 1 - \frac{R}{2} \quad \text{for} \quad R < 1 \]  

(Eq. 2.50a)

\[ F = \frac{1}{2R} \quad \text{for} \quad R \geq 1 \]  

(Eq. 2.50b)

The global orientation factor given by Eqs. 2.46, 2.50a and 2.50b is plotted in Figure 2.20 for the case of 35 mm long hooked-ended fibres assuming a hook length of 4.3 mm and snubbing length of 1.65 mm.

![Figure 2.20](image)

Figure 2.20 Plot of global orientation factor against displacement ratio for lumped bond shear stress approach.
The VEMII predictions, as given by Eqs. 2.39 and 2.46, are compared to the experimental results for the fibre volume tests of Lee and Foster (2006a). In the discrete fibre tests undertaken as a part of the same study for hooked-ended fibres with a diameter $d_f = 0.9$ mm and $l_f = 60$ mm it was determined that $\alpha_f = 0.0075$ (see Figure 2.1). Voo and Foster (2003) determined a value of $\alpha = 3d_f/l_f$ for their uniform bond model to describe Mode I fracture of fibre reinforced concrete. Keeping with this methodology, from the discrete fibre tests we obtain:

$$\alpha_{II} = 0.5 \frac{d_f}{l_f}$$  \hspace{1cm} (Eq. 2.51)

Eq. 2.51 is used in the model versus test comparisons that follow.

In Figure 2.21 the results for the lumped bond model are compared with the results for the hook-ended fibre tests of Lee and Foster (2006a). In these tests 35 mm long by 0.55 mm diameter fibres were used in the fibre ratios of $\rho_f = 0.005, 0.010, 0.015$ and 0.020. In the comparison, the length of the snubbing region is taken as 1.65 mm (that is, three times the diameter of the fibre) and the length of the hook, $l_{hook} = 4.3$ mm. The bond strengths were taken as those determined for the 60 mm long discrete fibre tests and given in Eqs. 2.6b and 2.10. A reasonable correlation is observed between the model and the test data for displacements up to 10 mm. Beyond $w_s = 10$ mm, the assumption that a proportion of the fibres pullout from the longer embedded side becomes more important.

![Figure 2.21 Hooked-ended fibre shear stress against crack sliding displacement for experimental and VEMII lumped bond stress model.](image-url)
In Figure 2.22 the results for the lumped model are compared for the 13 mm long by 0.2 mm diameter straight fibres for $\rho_f = 0.005, 0.010, 0.015$ and 0.020 with the length of the snubbing region taken as 0.6 mm, that is, three times the fibre diameter. The results show a poor correlation with the tests data. In Figure 2.23 the model predictions are plotted with the length of the snubbing zone taken as $l_{snub} = 1.65$ mm. In this case the correlation is improved and demonstrates the sensitivity of the model predictions for the straight fibres to the snubbing length and snubbing bond stress.

![Graph of Lumped Bond Model and Straight Fibres](image)

**Figure 2.22** Straight fibre shear stress against crack sliding displacement for experimental and VEMII lumped bond stress model.

![Graph of Lumped Bond Model and Straight Fibres](image)

**Figure 2.23** Straight fibre shear stress against crack sliding displacement for experimental and VEMII lumped bond stress model (constant snubbing length).
2.7.4 Uniform Bond Shear Stress Model

For the uniform bond shear stress approach, the force in a single fibre is (Voo and Foster, 2003):

\[
P_f = k \pi d_f \tau_{b,\text{ave}} \frac{l_f}{2}
= 2k A_f \tau_{b,\text{ave}} l_f / d_f
\]  
(Eq. 2.52)

where \( \tau_{b,\text{ave}} \) is the average bond stress along the fibre length and the local orientation factor, \( k \), is given by:

\[
k = 0 \quad \text{for } w_s < w_{es} \text{ and } w_s \geq l_u
k = 2(l_u - w_s)/l_f \quad \text{for } w_{es} \leq w_s < l_u
\]  
(Eq. 2.53a)
(Eq. 2.53b)

The shear stress contribution from the fibres is obtained from the force in a single fibre by integrating Eq. 2.52 over a plane of unit area as follows:

\[
\tau_f = 2 K_T A_f \tau_{b,\text{ave}} \frac{l_f}{d_f} \frac{P_f}{2A_f}
= K_T \tau_{b,\text{ave}} \rho_f l_f / d_f
\]  
(Eq. 2.54)

Again assuming a uniform distribution of fibres we obtain Eq. 2.47b. For the uniform bond model \( F = 1 \) and

\[
k_{\text{ave}} = \frac{1}{2} - \frac{w_s}{l_f}
\]  
(Eq. 2.55)

with \( N^{-1} \Sigma k(w_s) = k_{\text{ave}} p \) and substituting into Eq. 2.47b gives:

\[
K_T = k_{\text{ave}} p \left( \frac{1}{2} - \frac{\theta_{\text{crit}}}{\pi} \right)
\]  
(Eq. 2.56)

where \( p \) is given by Eq. 2.45.

Finally, substituting Eqs. 2.45 and 2.55 into Eq. 2.56, we obtain:
\[ K_f = \frac{1}{\pi} \tan^{-1} \left( \frac{w_s}{\alpha_{ll} l_f} \right) \left( 1 - \frac{2w_s}{l_f} \right)^2 \]  \hspace{1cm} \text{(Eq. 2.57)}

In their calibrations of the uniform bond approach, Voo and Foster (2003) determined that, for fibre reinforced mortars, the average bond stress could be taken as:

\[ \tau_{b,\text{ave}} = f_{ct} \quad \text{for straight fibres} \]  \hspace{1cm} \text{(Eq. 2.58a)}

\[ \tau_{b,\text{ave}} = 2f_{ct} \quad \text{for hooked-ended fibres} \]  \hspace{1cm} \text{(Eq. 2.58b)}

For the comparisons with the test data of Lee and Foster (2006a), the tensile strength is taken as equal to the measured shear strength for the unreinforced mortar, that is, \( f_{ct} = \tau_0 = 4.4 \text{ MPa} \). For the fibres, the average bond strengths are taken as:

\[ \tau_{b,\text{ave}} = \tau_0 = 4.4 \text{ MPa} \quad \text{for straight fibres} \]  \hspace{1cm} \text{(Eq. 2.59a)}

\[ \tau_{b,\text{ave}} = 2\tau_0 = 8.8 \text{ MPa} \quad \text{for hooked-ended fibres} \]  \hspace{1cm} \text{(Eq. 2.59b)}

In Figures 2.24 and 2.25a the predictions for the uniform bond approach are compared with the hooked-ended and straight fibre specimen tests of Lee and Foster (2006a), respectively. Again it is observed that the hooked-ended fibre predictions compare well with the test data while the straight fibre predictions are overly conservative. In Figure 2.25b the prediction for the straight fibres are compared against the test data taking the average bond stress as \( \tau_{b,\text{ave}} = 8.8 \text{ MPa} \), and in this case the correlation is improved.

![Graph showing the comparison of experimental and VEMII uniform bond stress models.](image)

Figure 2.24 Hooked-ended shear stress against crack sliding displacement for experimental and VEMII uniform bond stress model.

Behaviour of Steel Fibre Reinforced Mortar in Shear III: Variable Engagement Model II
Figure 2.25  Straight fibre shear stress against crack sliding displacement for experimental and VEMII uniform bond stress model:
(a) $\tau_{b,\text{ave}} = 4.4$ MPa; (b) $\tau_{b,\text{ave}} = 8.8$ MPa.

Comparing the general shapes of the curves for the uniform bond model (Figures 2.24 and 2.25b) and the lumped model (Figures 2.21 and 2.23), as expected the lumped models generally represent the response better, particularly for the straight fibres. This said, given the general experimental scatter, for this type of problem, the uniform bond approach performs adequately. More calibration and verification data is required, however, for the smaller straight fibres.
2.7.5 Fracture Energies

The fracture energies for the hooked ended and straight fibres, as measured as the area under the load versus CSD curves, used in the randomly distributed fibre tests and the different VEMII bond stress models are presented in Table 2.2. In this table, the coefficient of variation (COV) for the uniform model (with \( \eta_{b,ave} = 8.8 \text{ MPa} \)) is lower that for the lumped model (with \( l_{snub} = 1.65 \text{ mm} \)) and is indicative of a lower deviation in the fracture energies. However, the average for the lumped model is close to unity, which suggests that the fracture energies for this model are comparable to the experimental results. Generally, the fracture energies reflect the variations that exist between the VEMII calculations and the test data.

Table 2.2 Fracture energy from fibre and matrix contributions for lumped and uniform bond stress approaches and test data.

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Fracture Energy (N/mm)</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Exp.</td>
<td>Lumped Model ( l_{snub} = 1.65 \text{ mm} )</td>
<td>A/Exp.</td>
</tr>
<tr>
<td>DSR-H-0.5</td>
<td>10.2</td>
<td>12.5</td>
<td>1.23</td>
</tr>
<tr>
<td>DSR-H-1.0</td>
<td>22.3</td>
<td>23.5</td>
<td>1.05</td>
</tr>
<tr>
<td>DSR-H-1.5</td>
<td>30.1</td>
<td>34.5</td>
<td>1.14</td>
</tr>
<tr>
<td>DSR-H-2.0</td>
<td>49.7</td>
<td>45.6</td>
<td>0.93</td>
</tr>
<tr>
<td>DSR-S-0.5</td>
<td>6.3</td>
<td>4.7</td>
<td>0.75</td>
</tr>
<tr>
<td>DSR-S-1.0</td>
<td>9.9</td>
<td>8.0</td>
<td>0.81</td>
</tr>
<tr>
<td>DSR-S-1.5</td>
<td>13.6</td>
<td>11.3</td>
<td>0.83</td>
</tr>
<tr>
<td>DSR-S-2.0</td>
<td>16.7</td>
<td>14.6</td>
<td>0.87</td>
</tr>
<tr>
<td>Average</td>
<td>-</td>
<td>-</td>
<td>0.95</td>
</tr>
<tr>
<td>COV</td>
<td>-</td>
<td>-</td>
<td>0.17</td>
</tr>
</tbody>
</table>
2.8 Hybrid Fibre Reinforced Concrete

The randomly distributed fibre specimens described in this report consisted of steel fibres of the one type. The variable engagement modelling concept allows calculation of any number of different fibre types across the cracking surface. Hybrid fibre combinations generally include a mixture of two different fibre types which may be of diverse lengths (Mihashi and Kohno, 2007). For example, one fibre is a short straight fibre and the other one a longer hooked-ended fibre (Markovic et al., 2003, Varendra, 2007). In theory, the short length fibres are effective at the early stages of pullout and retard the propagation of crack growth along the cracking plane; and the longer fibres contribute after the short fibres are no longer effective in the later stages of pullout. Uniaxial tests show that there is an improvement in the ductility in the post-peak phase of loading (Markovic et al., 2003).

The VEMII can be adapted to include hybrid fibre arrangements by including the appropriate proportion of $\rho_f$ into the equations and summing the different fibre contributions. Hooked-ended ($l_f = 35 \text{ mm}$) and straight ($l_f = 13 \text{ mm}$) fibres are considered for the hybrid arrangements in the shear stress plots.

For the parametric analysis, 50 percent of short and 50 percent of long fibres are used with the lumped bond shear stress approach assuming constant snubbing length ($l_{snub} = 1.65 \text{ mm}$) for $\rho_f = 0.010$ and 0.020. The resulting plots are shown in Figure 2.26. From the figure, it is seen that the 35 mm long hooked-ended fibres dominate the shear behaviour and are the controlling fibre type in the hybrid combination. The replacement of half of the longer fibre with shorter fibres does not, in general, show an improved behaviour in shear. The differences between the improved response in tension, as reported by Markovic et al. (2003), and that of the theoretical behaviour in shear may lie in the generally better post-peak response observed in this study of the mortar component (Figure 2.18) as compared to that in tension. For example, Voo and Foster (2003) gave the attenuation factor for mortar in tension, see Eq. 2.37, as $c_2 = 30$. Thus, the early stages of the post-peak response are controlled by the mortar.
Figure 2.26 Shear stress against CSD for VEMII hybrid lumped bond stress model ($l_{sub} = 1.65$ mm): (a) $\rho_f = 0.010$; (b) $\rho_f = 0.020$ (note the hybrid is for 50% hooked and 50% straight fibres).
CHAPTER 3

CONCLUSIONS

In this study a model has been developed, viz VEMII, to describe the shear stress versus crack sliding displacement for a fibre reinforced mortar composite subjected to Mode II fracture. The model is developed by integrating the effects of individual fibres crossing a cracking plane, together with the plain mortar component.

In the development of the VEMII, the results and observations of discrete fibre tests were used, including visual data on the internal mechanisms obtained from gamma ray imaging, to assess the contribution of each component of the fibre to the total. To this end, four distinct regions were identified as contributing to the overall behaviour; viz, the hook, the straight and snubbing portions and fibre bridging component. Of these regions, the hook (if any) and the snubbing zone were found to dominate the bond behaviour and a lumped bond model was developed.

In the lumped bond approach, the bond is concentrated in two regions, at the hook and in the snubbing zone. The approach best fits the observations from the discrete fibre tests. In comparing the results of the model against experimental data the results were mixed. While the model was developed from individual fibre tests with hooked-ended and straight fibres of 0.9 mm diameter and 60 mm and 48 mm long, respectively, the shear tests were for fibre reinforced mortars with 0.55 mm diameter by 35 mm long hooked-ended fibres and 0.2 mm diameter by 13 mm long straight fibres. A comparison of the model results showed a reasonable comparison against test data for the end hooked fibre tests, considering the variability of such tests, but was very conservative for the smaller straight fibres. To identify the potential for the model to describe well the behaviour of these fibres, discrete fibre testing is needed using the smaller fibres to better describe the bond mechanism at the individual fibre level. In particular, the length and shape of the snubbing zone and the resulting bond stresses. Also, the influence of the fibres on improving the response of the unreinforced mortar needs further consideration.

In addition to the lumped bond approach, a uniform bond model was developed. The uniform bond approach is more typical of models that have been used historically and, while it does not describe test observations regarding the importance of the hook or the snubbing zone, it is a somewhat simpler and justifiable approach. In the uniform bond model the bond is taken as distributed uniformly along the fibre length.
and has been calibrated using a significant number of test data for Mode I fracture. Thus, while there is limited data for Mode II fracture, particularly for higher crack sliding displacements the Mode I fracture data can be used in combination with the variable engagement model II approach to obtain shear stress versus crack sliding displacement predictions for a range of mortar and concrete strengths for straight and hooked-ended fibres. As for the lumped approach, the uniform bond approach was found to give a reasonable correlation with the 0.55 mm diameter by 35 mm long hooked-ended fibre tests of this study but gave significantly conservative results for the smaller straight fibre data.

While the tests undertaken in this study go someway towards understanding the behaviour of fibre reinforced mortars and concrete in shear, much more testing is needed. Specifically test data is required for a range of concrete and mortar strengths and a range of fibre types and sizes. Lastly, data is needed to test the validity of the VEMII approach to describe the shear stress versus crack sliding response of hybrid-reinforced composites.
REFERENCES


APPENDIX A

Gamma Ray Images at Peak Load

The digitized gamma ray images captured either side of the peak load for the hooked-ended fibres at $+0^\circ$, $+30^\circ$ and $+60^\circ$ and straight fibre specimens at $+0^\circ$ and $+60^\circ$ are shown in Figures A.1 to A.8. The images are shown on their side to facilitate comparisons between adjacent images. In the figures the direction of the shearing action is indicated by hatched shaded arrows and the original shape of the fibre illustrated in dotted outline. The details for one fibre from each group are included in the figures to enable comparisons between the same specimens. The shapes of the fibres across the shearing planes were determined from observations.

When the shearing action and fibre orientation angle are more complimentary, pullout is more easily achieved, as can be seen in Figures A.1 and A.4 for $\theta = +60^\circ$. The straight fibre in Figure A.4 does not possess the same shear resistance as the hooked-ended fibre in Figure A.1 where the anchorage provided by the hooks supplies the significant shear resistance. As the angle becomes more negative, fibre bending becomes more pronounced and the exposed portion of the fibre across the shear plane more easily identified as can be seen Figures A.2, A.3, A.5, A.6, A.7 and A.8. Again anchorage provided by the hooks reveals greater movement adjacent to the shear plane.
Appendix A  Gamma Ray Images at Peak Load

Figure A.1  Gamma ray image for specimen NDSI+6011.3 indicating fibre details; (a) photograph P3; (b) photograph P4.
Figure A.2  Gamma ray image for specimen NDSI+0H1.3 indicating fibre details; (a) photograph P3; (b) photograph P4.
Figure A.3  Gamma ray image for specimen NDSI-60H1:3 indicating fibre details; (a) photograph P9; (b) photograph P10.

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Figure A.4  Gamma ray image for specimen NDSI+60SI:3 indicating fibre details; (a) photograph P2; (b) photograph P3.
Figure A.5  Gamma ray image for specimen NDSI+0S1:3 indicating fibre details; (a) photograph P2; (b) photograph P3.
Figure A.6  Gamma ray image for specimen *NDST-60S1:3* indicating fibre details; (a) photograph P5; (b) photograph P6.
Figure A.7  Gamma ray image for specimen NDSI+30H1:2 indicating fibre details; (a) photograph P2; (b) photograph P3.
Load per fibre = 0.56 kN
\[ w_v = 8.73 \text{ mm} \]

\[ \text{snubbing length} = 2.8 \text{ mm} \]
\[ \text{bridging length} = 5.6 \text{ mm} \]

Fibre shape at start of test

\[ \text{arc length} = 2.8 \text{ mm} \]

\[ 84^\circ \]

\[ w_h \approx 2.1 \text{ mm} \]

(a)

\[ \text{Peak load per fibre} = 0.59 \text{ kN} \]
\[ w_{v0} = 9.9 \text{ mm} \]

Load per fibre = 0.44 kN
\[ w_v = 10.22 \text{ mm} \]

\[ \text{snubbing length} = 3.5 \text{ mm} \]
\[ \text{bridging length} = 6.4 \text{ mm} \]

Fibre shape at start of test

\[ \text{arc length} = 3.5 \text{ mm} \]

\[ 7.5 \text{ mm} \]

\[ 27.1 \text{ mm} \]

\[ w_h \approx 1.2 \text{ mm} \]

(b)

Figure A.8  Gamma ray image for specimen NDSI-30H1:2 indicating fibre details; (a) photograph P6; (b) photograph P7.
APPENDIX B

Elastic Strength of a Steel Fibre Subject to Combined Axial Tension and Bending Actions

In this appendix the model based on linear elastic-brittle behaviour (Figure B.1) is developed. For an elastic-brittle material exhibiting a purely brittle failure mode; once the limiting value (or in this case, yield value) has been attained the material suddenly collapses and no further load carrying capacity is possible, so there is no ductile phase. This is a lower bound approach which implies that the true solution lies above the results predicted by this method.

![Stress-strain curve for fibre elastic-brittle behaviour.](image)

Figure B.1 Stress-strain curve for fibre elastic-brittle behaviour.

B.1 Elastic Stress-Strain Distribution

The different stress and strain distributions occurring through the fibre depth are shown in Figures B.2 to B.9. The distributions are arranged as follows:

- Figure B.2 Elastic stress distribution for a fibre subjected to combined tension.
- Figure B.3 Elastic strain distribution for a fibre subjected to combined tension.
- Figure B.4 Elastic stress distribution for zero resultant strain in extreme compressive fibre ($\bar{d} = 1$).
- Figure B.5 Elastic strain distribution for zero resultant strain in extreme compressive fibre ($\bar{d} = 1$).

Behaviour of Steel Fibre Reinforced Mortar in Shear III: Variable Engagement Model II
Appendix B  Elastic Strength of a Steel Fibre Subject To Combined Axial Tension and Bending  

**Actions**

Figure B.6  Elastic stress distribution for a fibre subjected to combined tension and bending.

Figure B.7  Elastic strain distribution for a fibre subjected to combined tension and bending.

Figure B.8  Elastic stress distribution for pure bending ($\bar{d} = 0$).

Figure B.9  Elastic strain distribution for pure bending ($\bar{d} = 0$).

In the figures the following terms are used:

- $a_e$ = resultant elastic stress-strain constant
- $C_e$ = elastic compression force in fibre due to bending
- $d_f$ = fibre diameter
- $d_0$ = distance between the neutral axis and fibre plastic centroid
- $\bar{d}$ = bending moment factor expressed as the ratio between $d_0$ and $d_f/2$
- $M_e$ = elastic fibre bending moment
- $NA$ = position of neutral axis
- $N_e$ = resultant elastic tensile force in fibre
- $T_{ae}$ = elastic axial tension force in fibre
- $T_e$ = resultant elastic tension force in fibre due to bending
- $z$ = distance from fibre centreline, positive above the fibre centreline and negative below the fibre centreline
- $z_{ce}$ = distance from fibre centreline to centroid of elastic compression force
- $z_{te}$ = distance from fibre centreline to centroid of elastic tension force
- $\varepsilon_{ae}$ = elastic axial strain
- $\varepsilon_{be}$ = elastic bending strain
- $\varepsilon_{ce}$ = resultant elastic compressive strain in fibre
- $\varepsilon_{fy}$ = yield strain of a steel fibre
- $\varepsilon_{te}$ = resultant elastic tensile strain in fibre
- $\sigma_{ae}$ = elastic axial stress
- $\sigma_{be}$ = elastic bending stress
- $\sigma_{ce}$ = resultant elastic compressive stress in fibre
- $\sigma_{fy}$ = yield tensile strength of a steel fibre
- $\sigma_{te}$ = resultant elastic tensile stress in fibre
- $\sigma_{ce}$ = elastic stress in fibre at position $z$
Figure B.2 Elastic stress distribution for a fibre subjected to combined tension.

Figure B.3 Elastic strain distribution for a fibre subjected to combined tension.

Figure B.4 Elastic stress distribution for zero resultant strain in extreme compressive fibre ($d = 1$).
Appendix B: Elastic Strength of a Steel Fibre Subject to Combined Axial Tension and Bending Actions

Figure B.5 Elastic strain distribution for zero resultant strain in extreme compressive fibre ($d = 1$).

Figure B.6 Elastic stress distribution for a fibre subjected to combined tension and bending.

Figure B.7 Elastic strain distribution for a fibre subjected to combined tension and bending.
Appendix B  Elastic Strength of a Steel Fibre Subject To Combined Axial Tension and Bending

Actions

\[ 0 \leq a_c \leq 1 \]

Axial Stress + Bending Stress = Resultant Stress

Figure B.8  Elastic stress distribution for pure bending ($\overline{d} = 0$).

\[ 0 \leq a_c \leq 1 \]

Axial Strain + Bending Strain = Resultant Strain

Figure B.9  Elastic strain distribution for pure bending ($\overline{d} = 0$).

From the above figures (Figures B.2 to B.9) the following relationships can be evaluated:

\[ \overline{d} = \frac{2d_0}{d_f} \]  
  \[ \text{Eq. B.1a} \]

\[ \overline{d} = \frac{a_c \varepsilon_{fy}}{\varepsilon_{be}} - 1 \]  
  \[ \text{Eq. B.1b} \]

\[ \sigma_{ae} = \frac{a_c \sigma_{fy}}{1 + \frac{\overline{d}}{d}} \]  
  \[ \text{Eq. B.2} \]

\[ \sigma_{be} = \frac{a_c \sigma_{fy}}{1 + \frac{\overline{d}}{d}} \]  
  \[ \text{Eq. B.3} \]

\[ \sigma_{te} = a_c \sigma_{fy} \left( \frac{\overline{d} - 1}{d + 1} \right) \]  
  \[ \text{Eq. B.4} \]
\[ \sigma_{ce} = -\sigma_{te} \]  
(Eq. B.5)

\[ \sigma_{ze} = a_e \sigma_{fy} \left( \frac{2 \zeta + \bar{d}}{d_f + \bar{d}} \right) \]  
(Eq. B.6)

\[ \varepsilon_{ae} = a_e \varepsilon_{fy} \left( \frac{\bar{d}}{d + 1} \right) \]  
(Eq. B.7)

\[ \varepsilon_{be} = \frac{a_e \varepsilon_{fy}}{d + 1} \]  
(Eq. B.8)

\[ \varepsilon_{te} = a_e \varepsilon_{fy} \left( \frac{\bar{d} - 1}{d + 1} \right) \]  
(Eq. B.9)

\[ \varepsilon_{ce} = -\varepsilon_{te} \]  
(Eq. B.10)

The range of values for the constant \( a_e \) lies between zero and one. For a value of zero there is no applied load however when \( a_e \) is equal to one the fibre reaches the limiting yield stress and brittle failure takes place. The term \( \bar{d} \) is the ratio between the position of the neutral axis and half the fibre diameter. It is designated the bending moment factor in this report and is an indicator of the position of the neutral axis and can be used to determine the level of bending moment present in the fibre.

The case where \( \bar{d} \) is greater than one is shown in Figures B.2 and B.3. It represents the condition when the fibre is subjected to combined tensile forces only; no compression forces exist during this stage. The resultant horizontal force is obtained by integrating \( \sigma_z \) at an elemental strip (Figure B.10) over the depth of the section to give:

\[ N_e = \frac{d_f}{2} \int_{-\frac{d_f}{2}}^{\frac{d_f}{2}} \sigma_{ze} dA \]  
(Eq. B.11a)
Figure B.10  Elemental strip of fibre.

$$\sqrt{d_f^2 - 4z^2}$$

$$\text{shaded area} = \int_{-d_f/2}^{d_f/2} \frac{2z\,dz}{1 + \frac{d}{d_f/2}} \sqrt{d_f^2 - 4z^2}$$

$$N_e = \frac{a_e \sigma_f y}{l + d/2} \int_{-d_f/2}^{d_f/2} \frac{2z\,dz}{d_f/2 + d} \sqrt{d_f^2 - 4z^2} \, dz$$

$$= \frac{a_e \sigma_f y}{12(l + d/d_f)} \left[ 6d \, z \, d_f \sqrt{d_f^2 - 4z^2} + 3d \, d_f^3 \tan^{-1}\left(\frac{2z}{\sqrt{d_f^2 - 4z^2}}\right) \right]$$

$$- 2d^2 \sqrt{d_f^2 - 4z^2} + 8z^2 \sqrt{d_f^2 - 4z^2} \right] \frac{d_f^2}{2}$$

$$= A_f \frac{d}{d_f} \frac{a_e \sigma_f y}{l + d/2}$$

(Eq. B.11b)

From Eq. B.11b it can be seen that as the neutral axis moves further away from the section, $d$ becomes larger and the value of $N_e$ approaches $a_e A_f \sigma_f y$, which is the case for pure tension.

The occurrence of $d$ equal to one (Figures B.4 and B.5) happens when there is zero resultant compressive strain occurring in the fibre. No bending is possible at this juncture but a small upward movement of the neutral axis instigates bending. The extreme axial and bending strains are equal, that is $\varepsilon_{ae} = \varepsilon_{be} = 0.5 \sigma_f y$ and the
resulting stresses are $\sigma_{ue} = \sigma_{be} = 0.5 \, a_e \sigma_{fy}$. The axial force can be obtained from Eq. B.11b to give:

$$N_e = 0.5 \, a_e \, A_f \, \sigma_{fy}$$  \hspace{1cm} (Eq. B.12)

The above axial force at the commencement of bending represents half the maximum tensile force possible.

For the case where $0 < \frac{d}{2} < 1$ the horizontal force is:

$$N_e = T_e - C_e$$

$$= a_e \sigma_{fy} \, A_f \, \frac{\bar{d}}{1 + \frac{\bar{d}}{d}}$$  \hspace{1cm} (Eq. B.13)

where

$$T_e = \int \sigma_{ze} \, dA$$

$$= \frac{1}{3} \, a_e \, \sigma_{fy} \, \left( \frac{d_f}{2} \right)^2 \left\{ 3 \bar{d} \left( \frac{\pi}{2} + \sin^{-1} \left( \bar{d} \right) \right) \right.$$

$$\left. + \sqrt{1 - \left( \bar{d} \right)^2} \left( 2 + \bar{d}^2 \right) \right\}$$  \hspace{1cm} (Eq. B.14)

and

$$C_e = -\int \sigma_{ze} \, dA$$

$$= -\left( \frac{d_f}{2} \right) \left\{ 3 \bar{d} \left( \frac{\pi}{2} - \sin^{-1} \left( \bar{d} \right) \right) \right.$$

$$\left. - \sqrt{1 - \left( \bar{d} \right)^2} \left( 2 + \bar{d}^2 \right) \right\}$$  \hspace{1cm} (Eq. B.15)

The horizontal force in Eq. B.13 is the same as the force calculated in Eq. B.11. A range of resultant elastic axial force to maximum force ratios for typical $a_e$ values is
shown in Figures B.11 and B.12 plotted against bending moment factor and strain ratio respectively.

The bending moment is evaluated from the following equation:

$$M_e = T_e z_{te} + C_e z_{ce}$$  \hspace{1cm} \text{(Eq. B.16)}

where

$$z_{te} = \frac{d_f}{8} \frac{\left[ 3 \left( \frac{\pi}{2} + \sin^{-1}\left( \frac{d}{\overline{d}} \right) \right) + \overline{d} \sqrt{1 - \left( \frac{d}{\overline{d}} \right)^2} \left( 5 - 2 \left( \frac{d}{\overline{d}} \right)^2 \right) \right]}{\left( \sqrt{1 - \left( \frac{d}{\overline{d}} \right)^2} \right) \left( 2 + \left( \frac{d}{\overline{d}} \right)^2 \right) + 3 \overline{d} \left( \frac{\pi}{2} + \sin^{-1}\left( \frac{d}{\overline{d}} \right) \right)}$$  \hspace{1cm} \text{(Eq. B.17)}

$$z_{ce} = \frac{d_f}{8} \frac{\left[ 3 \left( \frac{\pi}{2} - \sin^{-1}\left( \frac{d}{\overline{d}} \right) \right) + \overline{d} \sqrt{1 - \left( \frac{d}{\overline{d}} \right)^2} \left( 2 \left( \frac{d}{\overline{d}} \right)^2 - 5 \right) \right]}{3 \overline{d} \left( \frac{\pi}{2} - \sin^{-1}\left( \frac{d}{\overline{d}} \right) - \sqrt{1 - \left( \frac{d}{\overline{d}} \right)^2} \right) - \sqrt{1 - \left( \frac{d}{\overline{d}} \right)^2} \left( 2 + \left( \frac{d}{\overline{d}} \right)^2 \right)}$$  \hspace{1cm} \text{(Eq. B.18)}

\[\text{Figure B.11  Plot of axial tension ratios against bending moment factor for fibres with } a_e = 0.25, 0.50, 0.75 \text{ and } 1.0.\]
Figure B.12  Plot of axial tension ratio against strain ratio for fibres with $a_e = 0.25, 0.50, 0.75$ and $1.0$.

For the case where $\overline{d}$ is zero (Figures B.8 and B.9) there are no axial forces present ($\epsilon_a = 0$) and the fibre is subjected to pure bending ($\sigma_b = a_e \sigma_{fy}$). As the value of $\overline{d}$ (Eq. B.16) approaches zero the moment increases. The maximum moment is:

$$ (M_e)_{\text{max}} = \frac{a_e \pi d^3}{32} \sigma_{fy} $$

(Eq. B.19)

The moment ratios are plotted in Figure B.13 with the model interaction diagram shown in Figure B.14.
Figure B.13  Plot of moment ratios for fibres with $a_e = 0.25, 0.50, 0.75$ and $1.0$.

Figure B.14  Interaction diagram for fibres with $a_e = 0.25, 0.50, 0.75$ and $1.0$. 
The relationship between axial force and moment in Figure B.14 is linear and can be expressed as:

\[
\left[ \frac{N_e}{\sigma_{fy} A_f} \right]_{at \ a_e} + \left[ \frac{M_e}{\left( \pi d_f^3 \sigma_{fy} / 32 \right)} \right]_{at \ a_e} = 1 \quad \text{(Eq. B.20)}
\]

The interaction diagram for the elastic-brittle model in Section 2.5.3 ($c = 0.38$ and $\beta = 92$) with the range of applicable orientation angles noted is shown in Figure B.15.

Figure B.15 Interaction diagram for straight fibres (elastic-brittle model).
APPENDIX C

Plastic Strength of a Steel Fibre Subject To Combined Axial Tension and Bending Actions

The development of the model based on rigid-plastic behaviour (Figure C.1) is outlined in this appendix. A rigid-plastic material is either not stressed in any way or is at full yield once load has been applied. Theoretically an infinite deformation capacity (unlimited ductility) is assumed during the plastic stage. However failure occurs when the fracture strain, \( \varepsilon_{fu} \) is reached. This is an upper bound approach which implies that the true solution lies below the results predicted by this method.

![Stress-strain curve](image)

Figure C.1 Stress-strain curve for fibre rigid-plastic behaviour.

C.1 Plastic Stress-Strain Distribution

The plastic stress and strain distributions through the fibre depth are shown in Figures C.2 to C.5. The distributions are arranged as follows:

- **Figure C.2** Plastic stress distribution for a fibre subjected to combined tension and bending.
- **Figure C.3** Plastic strain distribution for a fibre subjected to combined tension and bending.
- **Figure C.4** Plastic strain distribution for pure bending (\( \bar{d} = 0 \)).
- **Figure C.5** Plastic strain distribution for zero resultant strain in extreme compressive fibre (\( \bar{d} = 1 \)).
In the figures the following terms are used:

\( a_p \) = resultant plastic stress-strain constant
\( C_p \) = resultant plastic compression force in fibre
\( M_p \) = plastic fibre bending moment
\( N_p \) = resultant plastic tensile force in fibre
\( T_p \) = resultant plastic tension force in fibre
\( z \) = distance from fibre centreline, positive above the fibre centreline and negative below the fibre centreline
\( z_{cp} \) = distance from fibre centreline to centroid of plastic compression force
\( z_{ip} \) = distance from fibre centreline to centroid of plastic tension force
\( \varepsilon_{ap} \) = plastic axial strain
\( \varepsilon_{bp} \) = plastic bending strain
\( \varepsilon_{cp} \) = resultant plastic compressive strain in fibre
\( \varepsilon_{fu} \) = ultimate tensile strain of a steel fibre
\( \sigma_{fy} \) = yield tensile strength of a steel fibre

![Stress Distribution Diagram](image)

Figure C.2 Plastic stress distribution for a fibre subjected to combined tension and bending.

![Strain Distribution Diagram](image)

Figure C.3 Plastic strain distribution for a fibre subjected to combined tension and bending.
The boundary strain distributions for the bending moment factor, \( \bar{d} \) equal to zero and one are shown in Figures C.4 and C.5 respectively.

For the case where \( \bar{d} \) is zero (Figure C.4) there are no axial forces present (\( \varepsilon_{ap} = 0 \)) and the fibre is subjected to pure bending. As the value of \( \bar{d} \) approaches zero the moment increases. The maximum moment is:

\[
(M_p)_{\text{max}} = \sigma_{fy} \frac{\pi d_f^2}{8} \frac{2d_f}{3\pi} 2
= \sigma_{fy} \frac{d_f^3}{6}
\]  

(Eq. C.1)

![Diagram of plastic strain distribution for pure bending (\( \bar{d} = 0 \)).](image)

Figure C.4 Plastic strain distribution for pure bending (\( \bar{d} = 0 \)).

![Diagram of plastic strain distribution for zero resultant strain in extreme compressive fibre (\( \bar{d} = 1 \)).](image)

Figure C.5 Plastic strain distribution for zero resultant strain in extreme compressive fibre (\( \bar{d} = 1 \)).
The case for $\bar{d}$ equal to one (Figure C.5) occurs when there is zero resultant compressive strain occurring in the fibre, no bending is possible at this stage. The extreme axial and bending strains are equal, that is $\varepsilon_{ap} = \varepsilon_{bp} = 0.5 \, a_p \varepsilon_{fu}$. The maximum axial force is:

$$\left( N_p \right)_{\text{max}} = \sigma_{fy} \, A_f$$  \hspace{1cm} (Eq. C.2)

The range of values for the constant $a_p$ lies between zero and one. For a value of zero there is no applied load however when $a_p$ is equal to one the fibre reaches the ultimate strain and brittle failure takes place.

From Figures C.2 to C.5 the relationships shown below are obtained:

$$\bar{d} = \frac{2 \, d_0}{d_f}$$  \hspace{1cm} (Eq. C.3a)

$$\bar{d} = \frac{a_p \, \varepsilon_{fu}}{\varepsilon_{bp}} - 1$$  \hspace{1cm} (Eq. C.3b)

The resultant plastic axial force is obtained from horizontal equilibrium in Figure C.2 as follows:

$$N_p = T_p - C_p$$
$$= \frac{2}{\pi} \sigma_{fy} \, A_f \left[ \bar{d} \sqrt{1 - (\bar{d})^2} \right.$$
$$\left. + \sin^{-1}(\bar{d}) \right]$$  \hspace{1cm} (Eq. C.4)

where
\[ T_p = \int_{-d_0}^{d_f} \sigma_{fy} \, dA \]
\[ = \sigma_{fy} \int_{-d_0}^{d_f} \sqrt{d_f^2 - 4z^2} \, dz \]
\[ = \sigma_{fy} \left[ \frac{z}{2} \sqrt{d_f^2 - 4z^2} + \frac{d_f^2}{4} \tan^{-1} \left( \frac{2z}{\sqrt{d_f^2 - 4z^2}} \right) \right]_{-d_0}^{d_f} \]
\[ = \sigma_{fy} \left( \frac{d_f}{2} \right)^2 \left[ \frac{\pi}{2} + \frac{d_f}{\sqrt{d_f^2 - 4\bar{d}^2}} \sin^{-1} \left( \frac{\bar{d}}{d_f} \right) \right] \quad \text{(Eq. C.5)} \]

and

\[ C_p = \int_{-d_0}^{d_f} \sigma_{fy} \, dA \]
\[ = \sigma_{fy} \int_{-d_0}^{d_f} \sqrt{d_f^2 - 4z^2} \, dz \]
\[ = \sigma_{fy} \left[ \frac{z}{2} \sqrt{d_f^2 - 4z^2} + \frac{d_f^2}{4} \tan^{-1} \left( \frac{2z}{\sqrt{d_f^2 - 4z^2}} \right) \right]_{-d_0}^{d_f} \]
\[ = \sigma_{fy} \left( \frac{d_f}{2} \right)^2 \left[ \frac{\pi}{2} - \frac{d_f}{\sqrt{d_f^2 - 4\bar{d}^2}} \sin^{-1} \left( \frac{\bar{d}}{d_f} \right) \right] \quad \text{(Eq. C.6)} \]

The plastic axial forces (Eq. C.4) to maximum force ratios for the fibres used in this study are shown in Figures C.6 and C.7 plotted against bending moment factor and strain ratio respectively.
Figure C.6  Resultant plastic axial force to maximum axial force ratio against bending moment factor for rigid-plastic model.

Figure C.7  Resultant plastic axial force to maximum axial force ratio against strain ratio for rigid-plastic model.
The bending moment is evaluated from the following calculations which are plotted in Figure C.8 with the interaction diagram shown in Figure C.9.

\[
M_p = T_p z_{lp} + C_p z_{cp} = \frac{d_f^3}{6} \sigma_{fy} \left( \sqrt{1 - \left(\frac{d}{d_f}\right)^2} \right)^3
\]  
(Eq. C.7)

where

\[
y_{lp} = \frac{-d_0}{d_f^2} \int_{-d_0}^{\frac{d_f}{2}} \frac{z \sigma_{fy} \sqrt{d_f^2 - 4z^2}}{\sqrt{d_f^2 - 4z^2}} \, dz
\]  
(Eq. C.8a)

\[
y_{lp} = \frac{-1}{12} \left( \sqrt{d_f^2 - 4z^2} \right)^3 \left[ \frac{d_f}{2} \right]_{-d_0}^{\frac{d_f}{2}} + \frac{d_f^2}{4} \tan^{-1} \left( \frac{2z}{\sqrt{d_f^2 - 4z^2}} \right) \left[ \frac{d_f}{2} \right]_{-d_0}^{\frac{d_f}{2}}
\]

\[
y_{lp} = \frac{d_f}{3} \frac{\left( \sqrt{1 - \left(\frac{d}{d_f}\right)^2} \right)^3}{\left[ \frac{\pi}{2} + \frac{d_f}{d} \sqrt{1 - \left(\frac{d}{d_f}\right)^2} + \sin^{-1}(\frac{d}{d_f}) \right]}
\]  
(Eq. C.8b)

and
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\[
y_{ep} = \frac{-d_0}{d_f} \frac{1}{2} \int \frac{\sqrt{d_f^2 - 4z^2} \, dz}{\sigma_{fy}}
\]

\[
= \left[ -\frac{1}{12} \left( \sqrt{d_f^2 - 4z^2} \right)^3 \right] d_0 \frac{d_f}{2}
\]

\[
= \left[ \frac{z}{2} \sqrt{d_f^2 - 4z^2} + \frac{d_f^2}{4} \tan^{-1} \left( \frac{2z}{\sqrt{d_f^2 - 4z^2}} \right) \right]^{d_0} \frac{d_f}{2}
\]

\[
= -\frac{d_f}{3} \left[ \frac{\pi}{2 - \bar{d} \sqrt{1 - (\bar{d})^2}} - \sin^{-1} \left( \frac{\bar{d}}{\sqrt{1 - (\bar{d})^2}} \right) \right]
\]

\[
\frac{M_p}{d_f^3 \sigma_{fy}/6}
\]

\[
\text{Figure C.8} \quad \text{Resultant plastic bending moments to maximum bending moment ratio for rigid-plastic model.}
\]

Behaviour of Steel Fibre Reinforced Mortar in Shear III: Variable Engagement Model II
Figure C.9  Plot of interaction diagram for rigid-plastic model.

The relationship between axial force and moment ratios in Figure C.9 can be approximated by:

\[
\left( \frac{N_p}{\sigma_f A_f} \right)^{2.1} + \frac{M_p}{(d_f^3\sigma_f/6)} = 1
\]  
(Eq. C.10)

The interaction diagram for the rigid-plastic model in Section 2.5.3 (c = 0.34 and \( \beta = 20 \)) with the range of applicable orientation angles noted is shown in Figure C.10.
Figure C.10  Interaction diagram for straight fibres (rigid-plastic model).
APPENDIX D

Elastic-Plastic Strength of a Steel Fibre Subject To Combined Axial Tension and Bending Actions

The model based on elastic-plastic behaviour (Figure D.1) is described in this appendix. For an elastic-plastic material once the yield strain ($\varepsilon_{fy}$) has been exceeded the stress distribution is no longer linear but remains at its yield value during the ductile phase until the fracture strain, $\varepsilon_{fu}$ is reached.

![Stress-strain curve for fibre elastic-plastic behaviour.](image)

Figure D.1 Stress-strain curve for fibre elastic-plastic behaviour.

From Figure D.1 it can be seen that there are four different stages that a fibre can exist:

(i) fibre strains are less than yield;
(ii) fibre strains are at yield;
(iii) fibre strains are greater than yield; and
(iv) fibre strains are at ultimate.

The first two stages have been covered in Appendix B for the elastic-brittle model investigation and the final stage was considered in Appendix C with the rigid-plastic model. The third stage will be examined in more detail in the sections which follow.
D.1 Elastic-plastic Stress-Strain Distribution

The different stress and strain distributions occurring through the fibre depth are shown in Figures D.2 to D.11. The distributions are arranged as follows:

Figure D.2 Elastic-plastic stress distribution for a fibre subjected to combined tension.
Figure D.3 Elastic-plastic strain distribution for a fibre subjected to combined tension.
Figure D.4 Elastic-plastic stress distribution for zero resultant strain in extreme compressive fibre ($\bar{d} = 1$).
Figure D.5 Elastic-plastic strain distribution for zero resultant strain in extreme compressive fibre ($\bar{d} = 1$).
Figure D.6 Elastic-plastic stress distribution for a fibre subjected to combined tension and bending (extreme tension face at yield).
Figure D.7 Elastic-plastic strain distribution for a fibre subjected to combined tension and bending (extreme tension face at yield).
Figure D.8 Elastic-plastic stress distribution for a fibre subjected to combined tension and bending (extreme tension and compression faces at yield).
Figure D.9 Elastic-plastic strain distribution for a fibre subjected to combined tension and bending (extreme tension and compression faces at yield).
Figure D.10 Elastic-plastic stress distribution for pure bending ($\bar{d} = 0$).
Figure D.11 Elastic-plastic strain distribution for pure bending ($\bar{d} = 0$).

In the figures the following terms are used:

\[ \alpha_{ep} = \text{resultant elastic-plastic stress-strain constant in tension region} \]
\[ \alpha_{ep} = \text{resultant elastic-plastic stress-strain constant in compression region} \]
\[ \alpha_{yit} = \text{ratio between elastic yield strain and ultimate strain} \]
\[ C_{ep} = \text{resultant elastic compression force in fibre due to bending (elastic-plastic model)} \]
\[ C_{ep} = \text{resultant plastic compression force in fibre due to bending (elastic-plastic model)} \]
\[ d_f = \text{fibre diameter} \]
\[ d_0 = \text{distance between the neutral axis and fibre plastic centroid} \]
\[ d_f = \text{distance between the tension yield strain and fibre plastic centroid} \]
\[ d_2 = \text{distance between the compression yield strain and fibre plastic centroid} \]
\[ \bar{d} = \text{bending moment factor expressed as the ratio between } d_0 \text{ and } d_f/2 \]
\( \bar{d}_1 \) = tension yield strain factor expressed as the ratio between \( d_1 \) and \( d_f/2 \)

\( \bar{d}_2 \) = compression yield strain factor expressed as the ratio between \( d_2 \) and \( d_f/2 \)

\( M_{ep} \) = elastic-plastic fibre bending moment

\( NA \) = position of neutral axis

\( N_{ep} \) = resultant elastic-plastic tensile force in fibre

\( T_{axp} \) = elastic-plastic axial tension force in fibre

\( T_{ep} \) = resultant elastic tension force in fibre due to bending (elastic-plastic model)

\( T_{eppl} \) = resultant plastic tension force in fibre due to bending (elastic-plastic model)

\( z \) = distance from fibre centreline, positive above the fibre centreline and negative below the fibre centreline

\( z_{cep} \) = distance from fibre centreline to centroid of elastic compression force (elastic-plastic model)

\( z_{ceppl} \) = distance from fibre centreline to centroid of plastic compression force (elastic-plastic model)

\( z_{ep} \) = distance from fibre centreline to centroid of elastic tension force (elastic-plastic model)

\( z_{eppl} \) = distance from fibre centreline to centroid of plastic tension force (elastic-plastic model)

\( e_{axp} \) = elastic-plastic axial strain

\( e_{bep} \) = elastic-plastic bending strain

\( e_{cep} \) = resultant elastic-plastic compressive strain in fibre

\( e_{fu} \) = ultimate tensile strain of a steel fibre

\( e_{ep} \) = resultant elastic-plastic tensile strain in fibre

\( \sigma_{axp} \) = elastic-plastic axial stress

\( \sigma_{bep} \) = elastic-plastic bending stress

\( \sigma_{cep} \) = resultant elastic-plastic compressive stress in fibre

\( \sigma_{fy} \) = yield tensile strength of a steel fibre

\( \sigma_{ep} \) = resultant elastic-plastic tensile stress in fibre

\( \sigma_{epc} \) = elastic compressive stress in fibre at position \( z \) (elastic-plastic model)

\( \sigma_{eppl} \) = elastic tension stress in fibre at position \( z \) (elastic-plastic model)
Figure D.2  Elastic-plastic stress distribution for a fibre subjected to combined tension.

Figure D.3  Elastic-plastic strain distribution for a fibre subjected to combined tension.
Figure D.4 Elastic-plastic stress distribution for zero resultant strain in extreme compressive fibre ($d = 1$).

Figure D.5 Elastic-plastic strain distribution for zero resultant strain in extreme compressive fibre ($d = 1$).
Appendix D  Elastic-Plastic Strength of a Steel Fibre Subject To Combined Axial Tension and Bending Actions

![Diagram](image)

**Figure D.6** Elastic-plastic stress distribution for a fibre subjected to combined tension and bending (extreme tension face at yield).

\[ a_{yu} \leq a_{ep} \leq 1 \]

![Diagram](image)

**Figure D.7** Elastic-plastic strain distribution for a fibre subjected to combined tension and bending (extreme tension face at yield).

![Diagram](image)

**Figure D.8** Elastic-plastic stress distribution for a fibre subjected to combined tension and bending (extreme tension and compression faces at yield).

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Appendix D  Elastic-Plastic Strength of a Steel Fibre Subject To Combined Axial Tension and Bending Actions

Figure D.9  Elastic-plastic strain distribution for a fibre subjected to combined tension and bending (extreme tension and compression faces at yield).

Figure D.10  Elastic-plastic stress distribution for pure bending ($\bar{d} = 0$).

Figure D.11  Elastic-plastic strain distribution for pure bending ($\bar{d} = 0$).
From the above figures (Figures D.2 to D.11) the following relationships can be derived:

\[
\overline{d} = \frac{2d_0}{d_f}
\]  \hspace{1cm} (Eq. D.1a)

\[
\overline{d} = \frac{\alpha_{ep} \varepsilon_{fu}}{\varepsilon_{bep}} - 1
\]  \hspace{1cm} (Eq. D.1b)

\[
a_{yu} = \frac{\varepsilon_{fy}}{\varepsilon_{fu}}
\]  \hspace{1cm} (Eq. D.2)

\[
\overline{d}_1 = \frac{2d_1}{d_f}
\]  \hspace{1cm} (Eq. D.3a)

\[
\overline{d}_1 = \frac{a_{yu}}{\alpha_{ep}}\left(\overline{d} + 1\right) - \overline{d}
\]  \hspace{1cm} (Eq. D.3)

\[
\overline{d}_2 = \frac{2d_2}{d_f}
\]  \hspace{1cm} (Eq. D.4a)

\[
\overline{d}_2 = \overline{d}_1 + 2\overline{d}
\]  \hspace{1cm} (Eq. D.4b)

\[
\alpha_{ep1} = \alpha_{ep}\left(\frac{1 - \overline{d}}{1 + \overline{d}}\right)
\]  \hspace{1cm} (Eq. D.5)

\[
\sigma_{aep} = \sigma_{fy}\left(\frac{\overline{d}}{\overline{d} + \overline{d}_1}\right)
\]  \hspace{1cm} (Eq. D.6)

\[
\sigma_{bep} = \frac{\sigma_{fy}}{\overline{d} + \overline{d}_1}
\]  \hspace{1cm} (Eq. D.7)

\[
\sigma_{tep} = \sigma_{fy}\left(\frac{\overline{d} - 1}{\overline{d} + \overline{d}_1}\right)
\]  \hspace{1cm} (Eq. D.8)

\[
\sigma_{zept} = \sigma_{fy}\left(\frac{\overline{d} + 2\overline{z}}{\overline{d} + \overline{d}_1}\right)
\]  \hspace{1cm} (Eq. D.9)

\[
\sigma_{zepc} = -\sigma_{zept}
\]  \hspace{1cm} (Eq. D.10)

\[
\varepsilon_{aep} = \alpha_{ep} \varepsilon_{fu}\left(\frac{\overline{d}}{\overline{d} + 1}\right)
\]  \hspace{1cm} (Eq. D.11)

\[
\varepsilon_{tep} = \alpha_{ep} \varepsilon_{fu}\left(\frac{\overline{d} - 1}{\overline{d} + 1}\right)
\]  \hspace{1cm} (Eq. D.12)

\[
\varepsilon_{cep} = -\varepsilon_{tep}
\]  \hspace{1cm} (Eq. D.13)
The range of values for the constants $\alpha_{ep}$ and $\alpha_{ep1}$ lies between $\alpha_{yu}$ and one. For a value equal to $\varepsilon_y/\varepsilon_{yu}$ there is no plastic zone and the fibre stresses are linear however when it is equal to one the fibre reaches the ultimate stress and brittle failure begins to take place. The explanation for the term $\bar{d}$ is given in Appendix B. The term $\bar{d}_1$ is the tension yield strain factor and is expressed as the ratio between $d_f$ and $d_f/2$. It is an indicator of the extent of the linear tensile stress range. A similar explanation applies to the compression yield strain factor $\bar{d}_2$.

The case where $\bar{d}$ is greater than one is shown in Figures D.2 and D.3. It represents the circumstance when the fibre is subjected to combined tensile forces only. The resultant horizontal force is obtained by integrating $\sigma_{zep}$ at an elemental strip (Figure B.10) over the depth of the section:

\[
N_{ep} = T_{ep} + T_{ep1}
= \frac{\sigma_{fy} d_f^2}{24 \left( \bar{d} + \bar{d}_1 \right)} \left[ \frac{3 \pi \left( 2 \bar{d} + \bar{d}_1 \right) - 6 \bar{d}_1 \sin^{-1} \left( \frac{\bar{d}_1}{\bar{d}} \right)}{2 + \left( \frac{\bar{d}_1}{\bar{d}} \right)^2} \right]
\]  
(Eq. D.14)

where

\[
T_{ep} = \int_{\frac{d_f}{2}}^{d_f} \sigma_{zep} \, dA
= \int_{\frac{d_f}{2}}^{d_f} \sigma_{fy} \frac{d_f}{d_1} \frac{2 \bar{z} + \bar{d}}{\bar{d}_1 + \bar{d}} \sqrt{d_f^2 - 4 \bar{z}^2} \, d\bar{z}
= \int_{\frac{d_f}{2}}^{d_f} \frac{\sigma_{fy} d_f^2}{24 \left( \bar{d}_1 + \bar{d} \right)} \left[ \frac{2 \sqrt{1 - \left( \frac{\bar{d}_1}{\bar{d}} \right)^2} \left\{ \bar{d}_1 \left( 3 \bar{d} + 2 \bar{d}_1 \right) \right\} - 2}{3 \bar{d} \left\{ 2 \sin^{-1} \left( \frac{\bar{d}_1}{\bar{d}} \right) + \pi \right\}} \right]
\]  
(Eq. D.15)

and
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\[
T_{epf} = \int_{d_1}^{d_f} \sigma_{fy} \, dA
\]
\[
= \int_{d_1}^{d_f} \sigma_{fy} \sqrt{d_f^2 - 4z^2} \, dz
\]
\[
= \frac{\sigma_{fy} \, d_f^2}{8} \left[ \pi - 2 \sin^{-1} \left( \frac{d_1}{d_f} \right) - 2 \frac{d_1}{d_f} \sqrt{1 - \left( \frac{d_1}{d_f} \right)^2} \right] \quad \text{(Eq. D.16)}
\]

The moment at the section arises from equilibrium considerations as follows:

\[
M_{ep} = T_{ep} \, z_{tep} + T_{epf} \, z_{tep} \quad \text{(Eq. D.17)}
\]

where \( T_{ep} \) and \( T_{epf} \) are given above and the centroidal distances are shown below:

\[
z_{tep} = \frac{\left( \int z \, dV \right)_{tep}}{\left( \int dV \right)_{tep}}
\]
\[
= \frac{d_f}{2} \left[ \frac{\sigma_{fy}}{d_f} \frac{z}{d_1} + 2 \frac{d}{d_1} \sqrt{d_f^2 - 4z^2} \, dz \right]
\]
\[
= \frac{2z}{d_f} \left( \frac{d}{d_f} \frac{d}{d_1} \right) + \frac{\sigma_{fy}}{d_f} \frac{\sigma_{fy}}{d_1} \sqrt{d_f^2 - 4z^2} \, dz \quad \text{(Eq. D.18a)}
\]
\[ z_{tep} = -d_f \left[ \frac{2\sqrt{1 - \left( \frac{d_1}{d} \right)^2} \left\{ 3 \frac{d_1}{d} \left[ 1 - 2 \left( \frac{d_1}{d} \right)^2 \right] \right\}}{8 \left( \frac{2^{1/2} - \frac{d_1}{d} \left( 3 \frac{d}{d_1} + 2 \frac{d_1}{d} \right) - 2 \right) + 3 \frac{d}{d_1} \left( 2 \sin^{-1} \left( \frac{d_1}{d} \right) + \pi \right) } - 3\pi \right] \]  

(Eq. D.18b)

and

\[ z_{tep1} = \frac{\left( \int z \, dV \right)_{tep1}}{\left( \int dV \right)_{tep1}} \]

\[ = \frac{d_f}{2} \int_{d_1}^{d_f} z \sigma_{fy} \sqrt{d_f^2 - 4z^2} \, dz \]

\[ = \frac{d_f}{2} \int_{d_1}^{d_f} \sigma_{fy} \sqrt{d_f^2 - 4z^2} \, dz \]

\[ = \frac{2 \, d_f \left( \sqrt{1 - \left( \frac{d_1}{d} \right)^2} \right)^3}{3 \left[ \pi - 2 \sin^{-1} \left( \frac{d_1}{d} \right) - 2 \frac{d_1}{d} \sqrt{1 - \left( \frac{d_1}{d} \right)^2} \right]} \]  

(Eq. D.19)

When the value of \( \overline{d} \) is less than one the following relationships apply:

\[ N_{ep} = T_{ep} + T_{tep1} - C_{ep} - C_{cep} \]  

(Eq. D.20)

\[ M_{ep} = T_{ep} z_{tep} + T_{tep1} z_{tep1} + C_{ep} z_{cep} + C_{ep1} z_{cep1} \]  

(Eq. D.21)

where \( T_{tep1} \) and \( z_{tep1} \) are given above and the remaining terms are shown below:
\[ T_{ep} = \frac{\sigma_{fy} d_f^2}{12 \left( \overline{d}_f + \overline{d} \right)} \left[ \sqrt{1 - \left( \overline{d}_1 \right)^2} \left\{ 3\overline{d} + 2 \overline{d}_1 \right\} \overline{d}_1 - 2 \right] + \sqrt{1 - \left( \overline{d} \right)^2} \left( \left( \overline{d} \right)^2 + 2 \right) + 3 \overline{d} \left\{ \sin^{-1} \left( \overline{d} \right) + \sin^{-1} \left( \overline{d}_1 \right) \right\} \] 

(Eq. D.22)

\[ C_{ep} = \frac{\sigma_{fy} d_f^2}{24 \left( \overline{d}_f + \overline{d} \right)} \left[ 3 \overline{d} \left\{ 2 \sin^{-1} \left( \overline{d} \right) - \pi \right\} + 2 \left\{ 2 + \left( \overline{d} \right)^2 \right\} \sqrt{1 - \left( \overline{d} \right)^2} \right] \]

for \( \overline{d}_2 \geq 1 \) 

(Eq. D.23)

\[ C_{ep} = -\frac{\sigma_{fy} d_f^2}{12 \left( \overline{d}_f + \overline{d} \right)} \left\{ \sqrt{1 - \left( \overline{d}_2 \right)^2} \left( \frac{2\left( \overline{d}_2 \right)^2 - 3\overline{d} \overline{d}_2}{2} \right) \right\} + \left\{ \left( \overline{d} \right)^2 + 2 \right\} \sqrt{1 - \left( \overline{d} \right)^2} + 3 \overline{d} \left\{ \sin^{-1} \left( \overline{d} \right) - \sin^{-1} \left( \overline{d}_2 \right) \right\} \]

for \( \overline{d}_2 \leq 1 \) 

(Eq. D.24)

\[ C_{ep1} = \frac{\sigma_{fy} d_f^2}{8} \left[ \pi - 2\overline{d}_2 \sqrt{1 - \left( \overline{d}_2 \right)^2} + 2 \sin^{-1} \left( \overline{d}_2 \right) \right] \]

(Eq. D.25)

\[ z_{tep} = d_f \frac{\sqrt{1 - \left( \overline{d}_1 \right)^2} \left( -3\overline{d}_1 \left\{ 1 - 2 \left( \overline{d}_1 \right)^2 \right\} - 8 \overline{d} \left\{ 1 - \left( \overline{d}_1 \right)^2 \right\} \right)}{8 + 3 \overline{d} \left\{ \sin^{-1} \left( \overline{d} \right) + \sin^{-1} \left( \overline{d}_1 \right) \right\} + \overline{d} \sqrt{1 - \left( \overline{d} \right)^2} \left\{ 5 - 2 \left( \overline{d} \right)^2 \right\} } \]

(Eq. D.26)

\[ z_{cep} = d_f \frac{\left[ 3 \pi - 6 \sin^{-1} \left( \overline{d} \right) + 2\overline{d} \sqrt{1 - \left( \overline{d} \right)^2} \left( 2 \left( \overline{d} \right)^2 - 5 \right) \right]}{8 + 3 \overline{d} \left\{ \sin^{-1} \left( \overline{d} \right) - \pi \right\} + \overline{d} \sqrt{1 - \left( \overline{d} \right)^2} \left\{ 2 + \left( \overline{d} \right)^2 \right\} } \]

for \( \overline{d}_2 \geq 1 \) 

(Eq. D.27)
\[
\begin{align*}
\zeta_{cep} &= \frac{d_f}{8} \left[ \frac{3 \left\{ \sin^{-1}(\bar{d}) - \sin^{-1}(\bar{d}_2) \right\}}{+ \sqrt{1 - (\bar{d}_2)^2} \left\{ \frac{3 \bar{d}_2 \left( 1 - 2(\bar{d}_2)^2 \right)}{+ 8 \bar{d} \left( (\bar{d}_2)^2 - 1 \right) \right\}} \right] \\
&\quad + \bar{d} \sqrt{1 - (\bar{d})^2} \left( 5 - 2(\bar{d})^2 \right) \right] \quad \text{for } \bar{d}_2 \leq 1 \quad (\text{Eq. D.28})
\end{align*}
\]

\[
\zeta_{cep} = \frac{2}{3} \left[ \frac{d_f}{\sqrt{1 - (\bar{d}_2)^2}} \right]^3 \left[ -\pi + 2 \bar{d}_2 \sqrt{1 - (\bar{d}_2)^2} \right] \quad (\text{Eq. D.29})
\]

From previous calculations the maximum force is:

\[
(N_{ep})_{\text{max}} = A_f \sigma_{fy} \quad (\text{Eq. D.30})
\]

The elastic-plastic axial force ratios for the fibres used in this study are shown in Figures D.12 and D.13 plotted against bending moment factor and strain ratio respectively.

The maximum moment can be obtained from Eq. D.21 with \( \bar{d} \) taken equal to zero.

\[
(M_{pe})_{\text{max}} = \frac{\sigma_{fy} d_f^3}{48 d_l} \left[ \bar{d}_1 \sqrt{1 - (\bar{d}_1)^2} \left\{ 5 - 2(\bar{d}_1)^2 \right\} \right] \quad (\text{Eq. D.31})
\]

The bending moment ratios are plotted in Figure D.14 with the interaction diagram in Figure D.15.
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Figure D.12  Plot of resultant axial force to maximum axial force ratio against bending moment factor for elastic-plastic model.

Figure D.13  Plot of resultant axial force to maximum axial force ratio against strain ratio for elastic-plastic model.
Figure D.14  Plot of moment ratios for elastic-plastic model.

Figure D.15  Interaction diagram for elastic-plastic model.
The relationship between axial force and moment ratios in Figure D.15 can be expressed as:

\[
\left( \frac{N_{ep}}{\sigma_{fy} A_f} \right)^{1.8} + \left( \frac{M_{ep}}{M_{ep \max}} \right)^{1.05} = 1 \quad \text{(Eq. D.32)}
\]

The interaction diagram for the elastic-plastic model in Section 2.5.3 (\(c = 0.35\) and \(\beta = 21\)) with the range of applicable orientation angles noted is shown in Figure D.16.

![Interaction diagram for straight fibres (elastic-plastic model).](image)

**Figure D.16** Interaction diagram for straight fibres (elastic-plastic model).