NUMERICAL ANALYSIS OF CONTINUOUS COMPOSITE BEAMS UNDER SERVICE LOADING

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by

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ABSTRACT

This paper presents a method for the service-load analysis of continuous composite beams. Both short-term and time-dependent analyses are carried out, in which cracking, creep and shrinkage of the concrete slab are considered. The time-dependent response of individual cross-sections is modelled using the Age-Adjusted Effective Modulus Method coupled with a relaxation procedure, and the lengthwise or longitudinal analysis of the member makes use of the force method of structural analysis. Both propped and unpropped construction may be modelled. The numerical solution requires iteration, but is suited to straightforward spreadsheet or conventional programming on a personal computer, on which the analysis is performed rapidly. The scope of the method is demonstrated in a simple example of the behaviour of a two-span beam with the same sustained loading that is cast propped or unpropped.

KEY WORDS

Composite construction, continuous beams, cracking, creep, force method, moment redistribution, shrinkage.
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1. Introduction

Probably the most common composite steel-concrete structural member is the composite tee-beam, which is comprised of a concrete slab connected to a steel I-section by stud shear connectors. Composite tee-beams find extremely widespread use in both buildings and highway bridges. The attraction of standard composite beams with effective shear connection is that the steel component is subjected predominantly to tensile stresses, and the concrete to compressive stresses, thereby making the most efficient use of both materials. For simply supported composite beams, the substantial advantages in strength and stiffness that can be gained with an effective shear connection are quantified in Oehlers and Bradford (1995, 1999).

Because composite beams make use of an extensive concrete component, they are subjected to the time-varying effects of creep and shrinkage. The ramifications of these effects on the deformations of simply supported beams are quite well-researched, with a very recent bibliography being given by Dezi et al. (1998). Controlling deformations under service loading is a serviceability limit state problem, and although serviceability analyses are usually based on linear-elastic assumptions, the analysis of composite beams is nonlinear owing to the cracking, creep and shrinkage of the slab (Bradford and Gilbert 1989).

Frequently, composite beams are either continuous over an internal support in bridges, or have a rigid connection to an interior column in buildings. In both cases, the beam has a negative bending region in which the concrete is subjected to tension and the steel primarily to compression. Although this reversal of the benign characteristics of the steel and concrete in negative bending reduces the efficiency of composite beams below those subjected to positive bending only, there are still both strength and stiffness advantages if continuous construction is used. Most research in continuous composite beams has concentrated on the strength limit state (Oehlers and Bradford 1995, 1999), where ductility provisions must be met for advantageous rigid-plastic analysis to be used. Similar research on beam-to-column connections (Leon 1998) has again been directed towards the strength limit state. The service-load analysis of continuous composite beams is not straightforward, firstly because predicting even the short-term
response requires a knowledge of the points of contraflexure which are not known \textit{a priori}, and secondly because of the effects of creep and shrinkage in the long-term analysis which cause a time-dependent redistribution of bending moment.

This paper presents a numerical method which is able to analyse the short-term and time-dependent behaviour of a two-span continuous composite beam at service load levels which do not cause yielding of the steel component. The spans may be of differing lengths, and subjected to sustained uniformly distributed or point loads whose position within each span is specified. Both propped and unpropped construction are considered, and recourse is made to the Age-Adjusted Effective Modulus Method (AEMM) to model the time-dependent response (Bazant 1972). This method was first applied to continuous beams by Gilbert and Bradford (1995), but their study was restricted to two beams with two equal spans cast propped with a uniformly distributed load throughout, which is merely a representation of a propped cantilever. The redundant interior support reaction is determined by making use of the flexibility method of structural analysis (Hall and Kabaila 1986) that involves an iteration scheme owing to the different flexural rigidities of the beam in positive and negative bending. The method leads to very rapid solutions on personal computers, and its accuracy is calibrated experimentally against carefully-conducted tests undertaken by the authors (Bradford and Gilbert 1999).

2. Cross-Section Analysis

2.1 General
In analysing the composite cross-section at service load levels, the following assumptions are made:

1. Plane sections remain plane with negligible interface slip, so that the strain diagram throughout the cross-section is linear.
2. The short-term response of the concrete is linear elastic in compression, and in tension prior to cracking.
3. The concrete in tension carries no stress after cracking, and the tensile strength \( f_t = 0.6\sqrt{f_c} \), where \( f_c \) is the 28 day characteristic cylinder strength of the concrete, and the strengths are in units of N/mm\(^2\).

4. The stress-strain relationship for the steel is linearly elastic in tension and compression, with elastic modulus \( E_s \).

5. Creep and shrinkage are treated using the AEMM constitutive model.

6. Compressive strains are positive, tensile strains are negative, and positive bending moment causes tension in the bottom fibres of the cross-section.

7. The same slab reinforcement exists in regions of positive and negative bending throughout the length of the beam.

2.2 Short-term analysis

The composite cross-section shown in Fig. 1 is transformed in accordance with the short-term modular ratio \( n = E_d/E_c \), where \( E_c \) is the short-term elastic modulus of the concrete, and the steel areas are converted into equivalent areas of concrete. Under the action of a moment \( M \) applied to the composite cross-section, the position of the neutral axis varies with time due to creep and shrinkage, so in both the short-term and time-dependent analyses the top fibre of the cross-section is selected as a suitable invariant reference axis, as was suggested by Gilbert (1988). The section properties used in the subsequent analysis (first and second moments of area) are calculated about this reference axis.

The short-term strain distribution is shown in Fig. 2, where the top fibre strain is \( \varepsilon_{ot} \) and the curvature is \( \rho \). It has been shown (Gilbert 1988) that the short-term axial force \( N_t \) and bending moment \( M_t \) (about the reference axis) are given by

\[
N_t = E_c \varepsilon_{ot} A - E_c \rho B \tag{1}
\]

\[
M_t = -E_c \varepsilon_{ot} B + E_c \rho I' \tag{2}
\]
in which \( A \) is the area of the cross-section, and \( B \) and \( I' \) are the first and second moments of area respectively of the transformed cross-section about the reference axis. Although the slab and steel joist contain axial actions, their net resultant is zero over the whole cross-section as the latter is subjected to pure bending only. Hence Eqs. 1 and 2 may be written in matrix form as

\[
E_c \begin{bmatrix}
  A & -B \\
  -B & I'
\end{bmatrix}
\begin{bmatrix}
  e_{oi} \\
  \rho_i
\end{bmatrix} =
\begin{bmatrix}
  0 \\
  M_i
\end{bmatrix}
\]  

(3)

which may be inverted to produce

\[
e_{oi} = \frac{BM_i}{E_c(AI' - B^2)}
\]  

(4)

\[
\rho_i = \frac{AM_i}{E_c(AI' - B^2)}
\]  

(5)

The solution of Eqs. 4 and 5 requires expressions for the transformed section properties. In the positive bending region, if the slab is not cracked then clearly

\[
A = b_2 D_c + nA_s + (n-1)A_r
\]  

(6)

\[
B = \frac{b_2 D_c^2}{2} + nA_s d_r + (n-1)A_r d_r
\]  

(7)

where \( d_r \) is the depth from the reference level to the centroid of the steel joist and, by definition, the depth of the neutral axis below the reference fibre \( d_n \) is

\[
d_n = \frac{B}{A}
\]  

(8)

If \( d_n \), so computed, is greater than \( D_c \), then the uncracked assumption is correct, and in addition to Eqs. 6 and 7,
\[ I' = \frac{b_r D_c^3}{3} + n (I_z + A_r d_r^2) + (n-1) A_r d_r^2 \]  

(9)

On the other hand, if \( d_n < D_c \) the concrete cracks in regions where the tensile stress exceeds \( f_t \). Strictly speaking, the position of the neutral axis if \( f_t \) is set as zero can be determined by applying simple linear elastic analysis (Oehler and Bradford 1995, 1999) as the positive root of the equation

\[ \frac{b_r d_n^2}{2} + (n-1) A_r (d_n - d_r) = n A_r (d_z - d_n) \]  

(10)

where the use of \((n - 1)\) in the reinforcement term assumes that the neutral axis lies below the reinforcement, which will almost always be the case in practice. Equation 10 may be solved for \( d_n \), and Eqs. 6, 7 and 9 used to determine the short-term section properties in positive bending. However, the tensile region of the slab in positive bending is reasonably close to the neutral axis. Because of this, even if the concrete tensile stress in this region exceeds \( f_t \) and is therefore ineffective, its close proximity to the neutral axis has only a minor effect on the flexural rigidity (Oehler and Bradford 1995, 1999), and it is sufficiently accurate (and slightly conservative) to simply take \( d_n \) as the root of Eq. 10.

In the negative bending region, provided the moment is less than the cracking moment \( M_{cr} \) that causes the top fibre stress to reach its tensile strength \( f_t \), then the section is uncracked and the section properties are given by Eqs. 6, 7 and 9. The short-term top fibre strain at the onset of cracking is \( \varepsilon_{oi} = f_t / E_c \), so that from Eq. 4

\[ M_{cr} = \frac{f_t (AI' - B^2)}{B} \]  

(11)

As the negative moment increases slightly above \( M_{cr} \), the concrete slab is partially cracked until the moment reaches a value, only slightly larger than \( M_{cr} \), when the stresses throughout the slab exceed \( f_t \) and the slab is fully cracked. Hence to simplify
the analysis and with little error, it is assumed that once $M_{cr}$ in Eq. 11 is exceeded, the section is fully cracked so becomes a 'steel' section comprising of the reinforcement and steel joist. This 'steel' section is still transformed according to the short-term modular ratio $n$, so that

$$A = n(A_r + A_y)$$

(12)

$$B = n(A_r a + A_y a)$$

(13)

$$I' = n(A_r a_r^2 + A_y a_y^2 + I_y)$$

(14)

2.3 Time-dependent analysis

In order to calculate the changes in top fibre strain $\Delta \varepsilon_o$ and curvature $\Delta \rho$ due to creep and shrinkage, the total sustained load is assumed conservatively to be applied at time $t = t_o$, and the relaxation method described in Gilbert (1988) is used in conjunction with the AEMM. This method is well-documented in Gilbert (1988), and is not described here. The method leads to the following expressions in matrix format

$$\begin{bmatrix} \Delta \varepsilon_o \\ \Delta \rho \end{bmatrix} = \frac{1}{\bar{E}_e(A_r I'_e - B_e)} \begin{bmatrix} I'_e & B_e \\ B_e & A_e \end{bmatrix} \begin{bmatrix} \Delta N \\ \Delta M \end{bmatrix}$$

(15)

in which $\bar{E}_e$ is the age-adjusted effective modulus given by (Gilbert 1988)

$$\bar{E}_e(t_o, t) = \frac{E_c(t_o)}{1 + \chi(t_o, t)\phi(t_o, t)}$$

(16)

and $\phi$ is the creep coefficient, $\chi$ is the aging coefficient, and where the transformed properties $A_e$, $B_e$ and $I'_e$ are the counterparts of $A$, $B$ and $I'$, but calculated from the transformed section according to the age-adjusted modular ratio.
\[ \bar{n}_c(t_w, t) = \frac{E_c}{E_c(t_w, t)} \]  

In Eq. 15, \( \Delta N \) and \( \Delta M \) are the so-called restraining actions that arise from the relaxation procedure. They can be written in matrix format as

\[
\begin{bmatrix}
\Delta N \\
\Delta M
\end{bmatrix} = \bar{E}_c \bar{\phi} \begin{bmatrix}
A_c & -B_c \\
-B_c & I'_c
\end{bmatrix} \begin{bmatrix}
\varepsilon_{ox} \\
\rho_l
\end{bmatrix} + \bar{E}_c \varepsilon_{sh} \begin{bmatrix}
A_c \\
-B_c
\end{bmatrix}
\]

where \( \varepsilon_{sh}(t) \) is the shrinkage strain, and \( A_c, B_c \) and \( I'_c \) are the area, first moment of area and second moment of area about the reference axis respectively of the uncracked portion of the concrete component, dependent on the neutral axis depth \( d_n \). Of course, in the negative bending region where the moment exceeds \( M_{cr} \), these concrete section properties are taken as zero.

In a statically indeterminate structure, such as the continuous beam considered in this paper, lengthwise changes in the top fibre strain and curvature produce deformations that must be compatible with the external restraints, which herein is a condition of zero deflection at the internal support. Hence, small increments in strain and curvature \( \delta \varepsilon_o \) and \( \delta \rho \) that must be added to those in Eq. 15 are developed in the time domain. The calculation of these increments that are dependent on the redundancy are given in the longitudinal analysis presented in the following section, and they result in a time-varying change of the support reactions.

3. Lengthwise Member Analysis

3.1 Flexibility method

Figure 3 shows a two-span continuous beam subjected to a sustained uniformly distributed load \( w \) and sustained concentrated loads \( P_{x1} \) and \( P_{x2} \) in spans 1 and 2, whose lengths \( L_1 \) and \( L_2 \) may be different. The ramifications of propped and unpropped construction techniques (Oehlers and Bradford 1995) on the analysis will be considered later.
In accordance with the flexibility method of structural analysis (Hall and Kabaila 1986), equilibrium must be satisfied when choosing a primary moment $M^o$ and a reactant moment $M^l$. Here the redundant internal support is removed in calculating $M^o$, so that the compatibility condition of zero deflection is also eliminated. A unit force is then applied at this internal support position, and this is used to determine the reactant moment $M^l$. Any bending moment $M$ that is a linear combination of $M^o$ and $M^l$ is an equilibrium system, i.e.

$$M = M^o + XM^l$$

(19)

where $X$ is a scalar. The unique value of $X$ is determined by imposing the compatibility requirement at the interior support.

Use of the flexibility method requires the real curvature $\rho$ (in the short-term or long-term) as a function of the real moment $M$ at all cross-sections in Eq. 19. This can be calculated from the flexural rigidity $EI$ using Eq. 5 (based on the invariance of reference axis) as

$$\rho = \frac{AM}{E_c(AL' - B^2)} = \frac{M}{EI} = \frac{M^o + XM^l}{EI}$$

(20)

so that

$$EI = \frac{E_c(AL' - B^2)}{A}$$

(21)

Compatibility is enforced by satisfying the work theorem, viz. that the internal work equals the external work. Because the external work is given as the product of the internal support reaction and movement, which of course is zero, then the internal work must also equal zero. This can be written as
\[
\int M^1 (\rho_i + \Delta \rho + \delta \rho) \, dx = 0 \tag{22}
\]

where \( \rho_i \) is the short-term curvature due to the external loads, \( \Delta \rho \) is an imposed curvature due to creep and shrinkage, as is \( \delta \rho \) which is a curvature increment due to creep and shrinkage that is necessary to maintain equilibrium and compatibility in the redundant beam. Substituting Eqs. 19 and 20 into Eq. 22 and rearranging produces

\[
X = - \frac{\int M^1 \frac{M^\circ}{EI} \, dx + \int M^1 (\Delta \rho + \delta \rho) \, dx}{\int M^1 \frac{M^1}{EI} \, dx}
\tag{23}
\]

where the integrations are carried out along the entire beam.

3.2 Primary and reactant moments

The primary moment diagram is determined by removing the internal support in Fig. 3. From simple statics, the primary reaction at the support A is

\[
R^a_A = P_{i1} + P_{i2} + \frac{w(L_1 + L_2)}{2} - \frac{P_{i1}a_1 + P_{i2}a_2}{L_1 + L_2}
\tag{24}
\]

and so the primary moment distant \( x \) from the support A is

\[
M^\circ(x) = R^a_A x - \frac{wx^2}{2} - (x-a_1)^\circ P_{i1}(x-a_1) - (x-a_2)^\circ P_{i2}(x-a_2)
\tag{25}
\]

where \( \langle \rangle \) represent Macaulay brackets.

An upward unit reactant force applied at the support B on the reactant beam produces a reaction at A of
\[ R_A^i = \frac{L_1}{L_1 + L_2} \]  

(26)

so that from statics the reactant moment distant \( x \) from support \( A \) is

\[ M^i(x) = R_A^i x - (x - L_1)^o (x - L_2) \]  

(27)

### 3.3 Short-term analysis

The short-term member analysis may be undertaken by invoking Eq. 23 with the second integrand in the numerator set to zero, as creep and shrinkage have not yet developed. The integrations must be performed over the entire length of the beam, and since generally the beam will be cracked over the interior support, \( EI \) is a function of \( x \). This is illustrated in Fig. 4, where the beam has a flexural rigidity \( EI_{sa} \) in the positive moment region and \( EI_{cr} \) in the portion of the negative moment region where the concrete has cracked. The value of \( EI_{sa} \) may also change a little due to cracking, but is shown as constant in Fig. 4.

Because the coordinates \( x = \zeta_1 \) and \( \zeta_2 \) in Fig. 4 that define the cracked negative region are not known initially, an iterative scheme must be adopted. Firstly, it is assumed that \( EI = EI_{sa} \) throughout, and the extent of the cracked region \( (\zeta_1 \leq x \leq \zeta_2) \) determined by equating \( M(x) \) to \(-M_{cr}\). The value of \( X \) is recalculated as

\[ X = \frac{1}{EI_{sa}^2} \left[ \int_0^{\zeta_1} M^i M^o dx + \int_{\zeta_1}^{\zeta_2} M^i M^o dx + \int_{\zeta_2}^{L_1 + L_2} M^i M^o dx \right] 
   \quad - \frac{1}{EI_{cr}} \left[ \int_0^{\zeta_1} M^i M^1 dx + \int_{\zeta_1}^{\zeta_2} M^i M^1 dx + \int_{\zeta_2}^{L_1 + L_2} M^i M^1 dx \right] 
\]  

(28)

Once \( X \) has been determined, new values of \( \zeta_1 \) and \( \zeta_2 \) can be recalculated and Eq. 28 is again used. The integrations were performed using Gaussian quadrature, and convergence on the value of \( X \) (and the corresponding negative cracked region) was rapid.
3.4 Long-term analysis

Under long-term sustained loads, concrete creep and shrinkage cause the bending moments to redistribute. Since the primary and reactant bending moment diagrams $M_p(x)$ and $M_r(x)$ respectively depend only on the loads and external geometry, they do not alter. However, the value of $X$ changes to $X_e$ in the time domain, and because of the summation of the constant and monotonic terms in Eq. 19 the shape of the total bending moment diagram changes (or redistribution takes place). The effects of creep and shrinkage increase the short-term curvature by an increment known as the imposed deformation. Substituting the values of $\Delta N$ and $\Delta M$ in Eq. 18 into Eq. 15 produces

$$\Delta \rho = \alpha + \beta M_i$$

(29)

where $M_i$ is the short-term moment acting on the cross-section and

$$\alpha = \left( \frac{B_e A_e - A_e B_e}{A_e I_e' - B_e^2} \right) \varepsilon_{sh}$$

(30)

$$\beta = \frac{1}{E_e (A_e I_e' - B_e^2)} \left[ \phi \left( \frac{A_e AB_e - B_e AB_e - B_e BA_e + I_e' AA_e}{E_e (A I_e' - B_e^2)} \right) \right]$$

(31)

Of course, the values of $\alpha$ and $\beta$ will have different values in the uncracked and cracked positive and cracked negative regions since the section properties in Eqs. 30 and 31 are different, as noted earlier.

In a statically determinate beam, the imposed deformations can develop freely, and are simply a simple addition to the initial deformations. On the other hand, as also noted previously, a statically indeterminate beam provides a restraint that limits the free variation of the deformations; the latter needing to be compatible at the redundant restraint. Hence at time $t$, the curvature at a cross-section will be given by
\[
\rho = (\rho_i + \delta\rho) + \Delta\rho
\]  \hspace{1cm} (32)

and since \(\rho_i\) is fixed by Eq. 5 and \(\Delta\rho\) by Eq. 29, \(\delta\rho\) must be calculated so that the compatibility requirement of Eq. 22 is enforced. This requirement causes a change in the bending moment from its short-term value acting at the centroid of the cross-section transformed according to the short-term modular ratio \(n\) to the value \(M_e\) acting at the centroid of the cross-section transformed according to the age-adjusted effective modular ratio \(\bar{n}_e\), inducing a curvature

\[
\rho_e = \frac{M_e}{(EI)_e}
\]  \hspace{1cm} (33)

where \((EI)_e\) is the flexural rigidity of the age-adjusted transformed cross-section, and so \(\rho_e = \rho_i + \delta\rho\). By using Eq. 23,

\[
X_e = -\frac{\int M^1 \frac{M^e}{(EI)_e} \, dx + \int M^e \Delta\rho \, dx}{\int M^1 \frac{M^e}{(EI)_e} \, dx}
\]  \hspace{1cm} (34)

Although the primary and reactant bending moment diagrams are known as a function of \(x\), \(\Delta\rho\) and \((EI)_e\) are different in the positive and negative cracked and uncracked regions, which are not known at the outset. A similar iterative scheme is again deployed to determine the values of \(\zeta_1e\) and \(\zeta_2e\) in the time domain, and hence to converge on the value of \(X_e\). The time-dependent variation of bending moment is thus

\[
M_e = M^e + X_e M^i
\]  \hspace{1cm} (35)

In the time domain, the top fibre strain \(\varepsilon_o\) at a cross-section will be given by

\[
\varepsilon_o = \varepsilon_{o_i} + \Delta\varepsilon_o + \delta\varepsilon_o = \varepsilon_{o_i} + \Delta\varepsilon_o
\]  \hspace{1cm} (36)
where $\varepsilon_{oe}$ is obtained similarly to Eq. 4, and noting that the axial force at the centroid of the age-adjusted transformed section is zero, as

$$
\varepsilon_{oe} = \frac{B_c M_e}{E_c \left( A_e I'_e - B_c^2 \right)}
$$

and $\Delta \varepsilon_o$ is determined using Eqs. 15 and 18 as

$$
\Delta \varepsilon_o = \frac{M_s}{\left( A_e I'_e - B_c^2 \right)} \left[ \frac{\varphi(-B_c B_e I_a + B_e I'_a A_e + I'_a A_e B_e - I'_e B_c A_e)}{E_c \left( AI'_e - B_c^2 \right)} \right] + \left[ \frac{I'_a A_e - B_c B_e}{A_e I'_e - B_c^2} \right] \varepsilon_{sh}
$$

which can be substituted with Eq. 37 into Eq. 36 to obtain the time-dependent top fibre strain. The time-dependent curvature, which is also needed to define the strain distribution in the cross-section in the time domain, can be obtained from Eqs. 29 to 33.

### 3.5 Propped and unpropped construction

In propped construction, the concrete slab hardens with closely spaced props preventing the development of any sizeable bending moment. After the props are removed, the self-weight plus the superimposed sustained service load (and any live loading) are resisted entirely by the composite nature of the cross-section. Hence in the preceding analysis, the total uniformly distributed load $w$ is the sum of the self weight of the concrete and steel components, plus any superimposed sustained load.

On the other hand, in unpropped construction the steel component is required to resist its own self-weight, plus that of the wet concrete, until the member becomes composite when the concrete achieves sufficient strength. Hence prior to the development of composite action, the steel and concrete are subjected to curvature, but stress is present in the steel and not in either the concrete or the reinforcement, as the wet concrete does not provide any restraint. Assuming linear elastic behaviour, superposition can be used (Oehlers and Bradford 1995), requiring the following two analyses.
Firstly, under wet concrete loading, the analysis presented for the short-term loading is used with \( w \) being the self-weight loading only. The flexural rigidity is merely \( E_s I_s \) throughout, since the concrete neither cracks, creeps nor shrinks, and iteration is not necessary to calculate \( X \) in Eq. 23 (in which the second integral in the numerator vanishes). This elementary application of the force method produces the variation of the ‘unpropped’ bending moment \( M_{unpr}(x) \), from which the corresponding variation of curvature is

\[
\rho_{unpr} = \frac{M_{unpr}}{E_s I_s}
\]

(39)

The analysis of the composite section subjected to a superimposed sustained load \( w = w_s \) is now undertaken using the same method as before for both short-term and long-term loading. The long-term analysis should include the self-weight loading, as once the beam becomes composite the time-dependent behaviour of slab must be included. The final short-term and long-term curvatures are merely the sum of those determined for the loading \( w_s \) using the nonlinear analysis and \( \rho_{unpr} \) in Eq. 39.

3.6 Deflections

Both the short-term and long-term deflections can be obtained by twice integrating the lengthwise variation of curvature, incorporating the appropriate boundary conditions. Hence

\[
v(x) = \frac{1}{2} \int_0^x \rho(x)dx \, dx
\]

(40)

with the boundary conditions \( v(0) = 0 \) (which is already satisfied by Eq. 40), \( v(L_1) = 0 \) and \( v(L_1 + L_2) = 0 \). A schematic representation of Eq. 40 is shown in Fig. 5, together with the latter two boundary conditions. These boundary conditions are needed to bring the function \( v(x) \) in Eq. 40 back to \( v = 0 \) at the supports, and the difference between the
straight lines from 0 to \(v(L_1)\) and \(v(L_1)\) to \(v(L_2)\) shown in Fig. 5 clearly represent the real variation of deflection within each span. It is worth noting that in evaluating the integral in Eq. 40, the variation of short-term or long-term moment is quadratic for the loading considered, and so the curvature is also quadratic if constant flexural rigidities are assumed in the appropriate positive and negative regions, and use of Simpson's rule for the double integration will yield an exact result. Of course, for the integration to be exact, the cracked negative region must be identified (as was done in the iterative process illustrated in Eq. 28), and the Simpson's rule integration carried out in both the positive and negative uncracked, and negative cracked regions.

4 Experimental Validation

A carefully-conducted experimental study of the time-dependent behaviour of two-span continuous composite beams cast propped has been reported (Bradford and Gilbert 1999). This paper presents all the measured material properties for the beam with two equal spans of length 5.8 m. The data given in the experimental paper was used as input in the present numerical method.

Figure 6 shows plots of the long-term deflections of the beam under self-weight loading only, and self-weight with an additional superimposed load, that were measured experimentally by Bradford and Gilbert (1999). Also shown in this figure are the deflections determined from the numerical method. The good agreement shown in Fig. 6 for both loading curves demonstrates the accuracy of the method.

5. Illustration

The numerical method developed above was programmed for a personal computer, and the solutions are obtained very rapidly. As an example of the application of the method, a two-span composite beam with the geometry and properties shown in Table 1 was analysed when constructed both propped and unpropped.
Figure 7 shows both the bending moment diagrams and deflected shapes for the propped beam under short-term and long-term loading. In the short-term, 9.6% of each span had cracked adjacent to the internal support, which increased to 15.5% after the moment redistribution associated with creep and shrinkage.

The corresponding variation of bending moment and deflection for the same beam when constructed unpropped is shown in Fig. 8. It can be seen that the maximum deflection is increased by 33% in the short-term and by 14% in the long-term when the beam is cast unpropped relative to that when it is cast propped. On the other hand, the cracked region of the span adjacent to the interior support is 6.7% of the span length in the short-term and 13.4% of the span length in the long-term. This illustrates the general conclusion that the negative cracked region will be less for beams cast unpropped than if cast propped, but the deflections in unpropped beams will be greater than those in the same beam cast propped.

6. Conclusions

A numerical method has been presented that can analyse the short-term and time-dependent response of two-span continuous composite tee-beams at service load levels. The beam may have spans of different length, and concentrated sustained loads positioned arbitrarily within each span, as well as a uniformly distributed sustained load. The cross-sectional analysis was carried out using the AEMM coupled with a relaxation procedure described elsewhere, and the longitudinal analysis of the one-fold redundant beam is performed using the flexibility method of structural analysis. The method is applicable to beams cast either propped or unpropped.

The method has been programmed, and convergence of any iterative scheme needed in the analysis is rapid, so that results can be obtained rapidly on a personal computer. The accuracy of the model was demonstrated by a comparison with test results reported elsewhere, while a short illustration of the application of the method to a beam cast propped and unpropped was given. This sample study illustrated the differences in the behaviour of propped and unpropped beams when subjected to service loads.
7. References


Table 1 Details of beam analysed in illustrative example

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_c$ (mm)</td>
<td>150</td>
</tr>
<tr>
<td>$b_c$ (mm)</td>
<td>1800</td>
</tr>
<tr>
<td>$d_r$ (mm)</td>
<td>75</td>
</tr>
<tr>
<td>$A_r$ (mm$^2$)</td>
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</tr>
<tr>
<td>$D_s$ (mm)</td>
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<tr>
<td>$A_s$ (mm$^2$)</td>
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</tr>
<tr>
<td>$I_s$ (mm$^4$)</td>
<td>$75 \times 10^6$</td>
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<tr>
<td>$L_1$ (mm)</td>
<td>15,000</td>
</tr>
<tr>
<td>$L_2$ (mm)</td>
<td>15,000</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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</thead>
<tbody>
<tr>
<td>$f_c$ (N/mm$^2$)</td>
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</tr>
<tr>
<td>$E_c$ (kN/mm$^2$)</td>
<td>28.6</td>
</tr>
<tr>
<td>$E_s$ (kN/mm$^2$)</td>
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<tr>
<td>$\varepsilon_{th}$</td>
<td>$300 \times 10^{-6}$</td>
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<tr>
<td>$\phi$</td>
<td>2.5</td>
</tr>
<tr>
<td>$\chi$</td>
<td>0.8</td>
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<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{s1}$ (kN)</td>
<td>50</td>
</tr>
<tr>
<td>$P_{s2}$ (kN)</td>
<td>50</td>
</tr>
<tr>
<td>$a_1$ (mm)</td>
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<tr>
<td>$a_2$ (mm)</td>
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<tr>
<td>$w_{sustained}$ (kN/m)</td>
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</tr>
<tr>
<td>$w_{self\ weight}$ (kN/m)</td>
<td>7.44</td>
</tr>
</tbody>
</table>
Figure 1  Cross-section of composite T-beam
Figure 2 Short-term strain distribution
Figure 3 Two-span continuous composite beam
Figure 4 Longitudinal variation of flexural rigidity
Figure 5 Schematic representation of deflection calculation
Figure 6 Comparison of deflections with theory and tests
Figure 7 Propped beam results
Figure 8 Unpropped beam results

(a) bending moment diagram

(b) deflections