STRENGTH OF REINFORCED
CONCRETE SLENDER COLUMNS

B. V. RANGAN

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STRENGTH OF REINFORCED CONCRETE
SLENDER COLUMNS

by

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SYNOPSIS

A method for calculating the strength of reinforced concrete slender columns is presented. It is based on a simplified stability analysis of a standard pin-ended column. The method includes the creep deflection due to sustained loads as an additional eccentricity. The strengths calculated by the proposed method show good correlation with 80 test results reported in the literature.

When compared with the proposed method, the slender column provisions contained in the ACI Building Code are found to be conservative. Based on the method, proposals for design are made and illustrated by an example.

KEYWORDS

Columns; deflection; design; reinforced concrete; slenderness; strength.
INTRODUCTION

Reinforced concrete columns are usually classified as short and slender depending upon the order of magnitude of second order moments caused by the lateral deflection of the column and the lateral drift of the structure as a whole. For design purposes, the importance of second order moments is customarily related to the slenderness ratio, $L_e/r$, of the column where $L_e$ is the effective length and $r$ is the radius of gyration of the gross concrete cross-section. In the ACI Building Code (Ref.1), Section 10.11.4 provides the limits on $L_e/r$ below which the slenderness effects may be neglected.

The strength of a slender column is affected by many factors such as column length, end restraint conditions, distribution of bending moment, level of axial thrust, creep of concrete and bracing condition of the column. In design, the strength of a slender column is usually calculated using the following steps (Ref.2):

(i) Estimate the end restraint conditions and calculate the effective length of the column.

(ii) Where the slenderness effects cannot be neglected, replace the restrained column by an equivalent standard pin-ended column with equal end eccentricities of axial thrust at each end.

(iii) Convert the slender pin-ended column into an equivalent short column with appropriately magnified factored moment. The strength of the slender column is taken as the cross-sectional strength of this equivalent short column.

The design provisions contained in Section 10.11 of the ACI Code follow the above procedure. In this procedure, the strength of a standard pin-ended column plays an important role. Therefore, studies on slender columns have focussed attention on the behaviour and the strength of this standard column.

Several methods of stability analysis of slender column have been reported in the literature (Ref.3–8). Most of these methods tend to be complex for everyday design office use. On the other hand, the ACI Code provisions are simple to use but appear to be largely empirical. The aim of this paper is to present a method for calculating the
strength of slender columns. The proposed method is more rational and, at the same time, simple to use in routine design calculations.

**SIMPLIFIED STABILITY ANALYSIS**

Consider a standard pin-ended slender column as shown in Fig. 1. The axial load capacity of the column can be computed using a stability analysis. A pre-requisite for performing such an analysis is the knowledge of moment-thrust-curvature relations for the column section. By assuming a suitable function to express the lateral deflection in terms of the curvature at mid-height of the column, the moment-thrust-curvature relations can be converted to moment-deflection curves for a chosen value of the axial thrust. These curves then form the basis of the stability analysis. In Ref. 2, the procedure for carrying out such an analysis is illustrated in a graphical form. The calculations are iterative and can be done with the help of a computer.

In practice, a significant part of the design load applied on a column is sustained for a long period of time. The sustained load produces creep deflection, \( \Delta_{cp} \) (Fig.1) which causes a loss in load carrying capacity of the column.

In the simplified stability analysis proposed here, the creep deflection, \( \Delta_{cp} \) is treated as an additional eccentricity. Such an approach is used in the German Concrete Structures Code, DIN1045-1984 (Ref.9). The moment-thrust-curvature diagrams are approximated by elastic-plastic behaviour where \( P_{\text{max}} < \phi P_b \) or elastic-brittle behaviour where \( P_{\text{max}} \geq \phi P_b \). Assume a sine function for the deflection \( v(x) \):

\[
v(x) = \Delta \sin \left( \frac{\pi x}{L_e} \right)
\]

where \( \Delta \) is the deflection at the mid-height and \( L_e \) is the effective length of the column. Then the curvature is given by

\[
\frac{d^2v}{dx^2} = \frac{\pi^2}{L_e^2} \Delta \sin \left( \frac{\pi x}{L_e} \right)
\]

and hence at mid-height, where \( x = \frac{L_e}{2} \), the curvature \( \kappa \) is given by
\[ \kappa = (\pi^2/L_c^2) \Delta \]  

(3)

For a given column and for a chosen value of the axial thrust, the moment-curvature diagram is first converted to the moment-deflection diagram by means of Eq.(3). When the axial thrust is equal to the capacity of column, \( P_{\text{max}} \), the moment-deflection curve is taken as either \( OYY' \) (for \( P_{\text{max}} < \phi P_b \)), or \( OYY'' \) (for \( P_{\text{max}} \geq \phi P_b \)), see Fig. 2. If \( \kappa_y \) and \( \Delta_y \) are the curvature and the deflection respectively corresponding to the maximum moment, \( M_{\text{max}} \), stability failure is assumed to occur when the radial line \( AYB \) of slope equal to \( P_{\text{max}} \) passes through the point \( Y \) of the \( M-\Delta \) diagram (Fig. 2).

The axial load capacity of the column, \( P_{\text{max}} \), is then given by

\[ P_{\text{max}} = \frac{M_{\text{max}}}{(e + \Delta_{\text{cp}} + \Delta_y)} \]  

(4)

Creep Deflection

To perform the analysis, we need a suitable method to calculate \( \Delta_{\text{cp}} \). Elsewhere (Ref.10), the author has developed a simple expression for estimating the deflection of slender columns under sustained loads. Accordingly, the final total deflection, \( \Delta_{\text{tot}} \) at the mid-height of a standard pin-ended column is given by

\[ \Delta_{\text{tot}} = e/[(P_c/P_\phi)-1] \]  

(5)

where

\[ P_c = \pi^2 E I / L^2 \]  

(6)

\[ EI = \lambda E_c I_g / (1 + 0.8 \phi_{cc}) \]  

(7)

and

\[ \lambda = 0.6 + (e_b/8e) \]

\[ \leq 1.0 \]  

(8)

In these expressions, \( P_\phi \) is the axial thrust due sustained loads, \( \phi_{cc} \) is the creep factor,
and \( e_b \) is the value of \( e \) corresponding to balanced failure conditions and is equal to \( M_b/P_b \). Note that \( e_b \) depends on the column cross-section and is readily known from the usual strength calculations.

The total deflection is taken as the sum of the elastic component, \( \Delta_e \) and the creep deflection, \( \Delta_{cp} \). It is assumed that the shrinkage effects on the deflection are negligible. From Eq.(5), we can write

\[
\Delta_e = e/[P_{co}/P_c-1] 
\]

where

\[
P_{co} = \frac{\lambda \pi^2 E_e I_g}{L_e^2} 
\]

That is, \( \Delta_e \) is the particular value of \( \Delta_{tot} \) when \( \phi_{cc} = 0 \). Therefore, the creep deflection is

\[
\Delta_{cp} = \Delta_{tot} - \Delta_e 
\]

The deflections calculated by Eq.(5) show good correlation with the test values reported by Kordina (Ref.7), the mean value of calculated/test is 1.03 with a coefficient of variation of 14 percent.

**Calculation Procedure**

For a given column, the calculations may be performed as follows:

(i) Guess a suitable value for \( P_{\text{max}} \).

(ii) Where \( P_{\text{max}} \geq \phi P_b \), take \( P_{\text{max}} = \phi P_n \) and determine \( \phi M_n (-M_{\text{max}}) \) from the usual strength analysis of the cross-section. This calculation also gives the depth of the neutral axis, \( d_n \). In this case, the moment-curvature diagram is assumed to be elastic-brittle (line OYY" in Fig.2) and \( \kappa_y \) is taken as 0.003/d_n.
The deflection $\Delta_y$ is then given by Eq.(3).

(iii) Where $P_{\text{max}} < \phi P_b$, take $P_{\text{max}} = \phi P_y$ where $\phi P_y$ is the axial thrust acting on the section when the tensile steel just yields, i.e., the strain in the fibre at a distance $d$ from the compression face is $\varepsilon_{sy}$. The co-existing bending moment, $\phi M_y$, is equal to $M_{\text{max}}$. The values of $\phi P_y$ and $\phi M_y$ are obtained from the usual cross-sectional analysis by assuming a parabolic distribution for the concrete stresses in the compression zone. In this case, the moment-curvature diagram is elastic-plastic (line OYY' in Fig.2) and $\kappa_y$ is equal to $\varepsilon_{sy}/(d-d_n)$. The deflection $\Delta_y$ is, once again, given by Eq.(3).

(iv) For a known value of $P_{\phi}$, calculate $\Delta_{cp}$ by Eqs. (5), (9) and (11).

(v) For these values of $M_{\text{max}}$, $\Delta_y$, and $\Delta_{cp}$, calculate $P_{\text{max}}$ by Eq.(4). Use this new value of $P_{\text{max}}$ in step (i) and repeat steps (ii) or (iii), (iv) and (v) until the results converge.

**COMPARISON WITH TEST RESULTS**

The simplified stability analysis proposed in the previous section was used to calculate the axial load capacity of numerous test columns reported in the literature (Ref. 5–7). The results are presented in Tables 1 to 4. In all these calculations, the strength reduction factor was taken as unity.

In Table 1, the calculated and the test strengths are compared for 46 columns tested by Goyal and Jackson (Ref.6). The mean value of test/calculated is 1.02 with a coefficient of variation of 10 percent.

Table 2 presents the correlation between the calculated and the test strengths for 12 columns reported by Kordina (Ref.7). The mean value of test/calculated is 1.08 with a coefficient of variation of 15 percent.
In Table 3, the comparison of test and calculated strengths of 22 columns tested by Ramu et al (Ref.5) is given. For these data, the test/calculated ratio has a mean value of 1.14 with a coefficient of variation of 13 percent. Thirteen other columns tested by these investigators had e/h < 0.05 and two other columns had A_s'/A_s < 0.2. These results are not included in Table 3. For those unusual columns, test/calculated ratio is 1.35 with a coefficient of variation of 27 percent.

Table 4 presents a summary of correlation between test and calculated strengths. The test data covered a wide range of parameters: A_s'/A_g = A_s'/A_g  = 0.50, 0.85, 1.10, 1.20, 1.60, 2.15 percent; e/h = 0.05, 0.10, 0.15, 0.167, 0.20, 0.25, 0.33, 0.375, 0.5, 1.0; and L_e/r = 50, 80, 100, 120, 150. For a total of 80 columns, the mean value of test/calculated is 1.06 with a coefficient of variation of 13 percent.

**COMPARISON WITH ACI CODE**

In order to compare the simplified stability analysis presented herein with the provisions of Section 10.11 of the ACI Code (Ref.1), the following data were assumed: Rectangular column section, combined axial compression and uniaxial bending, standard pin-ended column, three different values of eccentricity, e, defined by e/h = 0.15, 0.30, and 0.60, f'_c = 32 MPa (4640 psi), f'_sy = 400 MPa (58 ksi), \( \beta_d = 0.7 \), A_s = A_s' = 0.01 A_g, d/h = 7/8, d'/h = 1/8, E_c = 28500 MPa (4133 ksi), \( \phi_{cc} = 2.5 \), and L_e/r = 25, 50, 75, and 100. For each column, the axial load capacity P_u (=\( \phi P_n \)) was calculated in accordance with the ACI Code (Ref.1) and P_\phi due to sustained loads was taken as P_{max}^{1/2}.

The results are presented in Fig. 3. It is seen that the ACI Code values are always conservative. The axial load capacity calculated by the ACI Code provisions is significantly smaller than that computed by the simplified stability analysis for columns with larger L_e/r ratios and smaller values of e/h. This may be due to the conservative EI value adopted by the Code in the calculation of the critical load.
DESIGN PROPOSALS

For design purposes, $P_{\text{max}}$ in Eq.(4) is replaced by the factored axial load, $P_u$ at an equivalent eccentricity, $e$. Then $M_{\text{max}}$ is the co-existing magnified factored moment, $M_c$. Eq. (4) can therefore be expressed as

$$M_c = P_u e (1 + \frac{\Delta_{cp} + \Delta_y}{e})$$  \hspace{1cm} (12)

or

$$M_c = P_u e \delta_b$$  \hspace{1cm} (13)

where the moment magnifier, $\delta_b$ is given by

$$\delta_b = 1 + (\frac{\Delta_{cp} + \Delta_y}{e})$$  \hspace{1cm} (14)

Once $P_u$ and $M_c$ have been calculated, the design of a slender column proceeds in the same manner as for a short column.

In Eq. (14), the creep deflection $\Delta_{cp}$ is given by Eq. (5), (9) and (11).

During the analyses of test results, it was consistently observed that the $\phi P_n$ versus $\Delta_y$ curve can be reasonably approximated by two straight–lines. That is, one straight–line between the points $(\phi P_n = \phi P_0, \Delta_y = 0)$ and $(\phi P_n = \phi P_b, \Delta_y = \Delta_{yb})$; and a second straight–line between the points $(\phi P_n = \phi P_b, \Delta_y = \Delta_{yb})$ and $(\phi P_n = 0, \Delta_y = \Delta_{yo})$. Here $\Delta_{yb}$ is the deflection at balanced failure conditions in combined axial compression and bending, and $\Delta_{yo}$ is the deflection in pure bending when the tensile steel just yields.

At balanced failure conditions, the curvature is $(0.003 + \varepsilon_{sy})/d$. From Eq. (3), therefore, $\Delta_{yb}$ becomes
\[ \Delta_{yb} = (0.003 + \varepsilon_{sy})L_e^2/\pi^2d \] (15)

For rectangular column sections in pure bending, the stress distribution in the compression zone concrete can approximately be taken as linear when the tensile steel just yields. Assuming that the internal lever arm between the concrete compressive force and the steel tensile force is equal to 0.875d, from linear elastic theory we obtain the depth of neutral axis as 0.375d. Therefore, the curvature \( \varepsilon_{sy}/(d-0.375d) = 1.6 \varepsilon_{sy}/d \) and from Eq.(3), \( \Delta_{yo} \) is given by

\[ \Delta_{yo} = 1.6 \varepsilon_{sy}L_e^2/\pi^2d \] (16)

If we approximate the \( \phi P_n \) versus \( \Delta_y \) curve by two straight-lines as mentioned above and replace \( \phi P_n \) by \( P_u \) we obtain the following expressions for \( \Delta_y \):

For \( P_u \geq \phi P_b \):

\[ \Delta_y = \Delta_{yb} (\phi P_o - P_u)/(\phi P_o - \phi P_b) \] (17)

For \( P_u \leq \phi P_b \):

\[ \Delta_y = \Delta_{yo} + (\Delta_{yb} - \Delta_{yo}) (P_u/\phi P_b) \] (18)

Based on the above, the following steps are proposed for the design of slender columns:

(i) Select a trial cross-section for the column. Calculate the effective length, \( L_e \) of the column by Section 10.11.2 of the ACI Code (Ref. 1).

(ii) Calculate the eccentricity, \( e \) for the equivalent standard pin-ended column from

\[ e = C_m (M_{2b}/P_u) \] (19)
where the factor \( C_m \) is given by the ACI Code Eq. (10–12) and \( M_{2b} \) is the value of the larger factored end moment.

(iii) Calculate \( \Delta_{cp} \) from Eqs. (5),(9), and (11) and, \( \Delta_y \) from Eq. (17) or (18). For these values of \( e, \Delta_{cp}, \) and \( \Delta_y \), calculate the moment magnifier, \( \delta_b \) by Eq. (14).

Check whether the design strength of the column cross-section is adequate to resist the combined effect of the factored actions \( P_u \) and \( M_{c'} \), where \( M_{c'} \) is given by Eq.(13).

The above design method is illustrated by the following example.

**EXAMPLE**

Check the adequacy of a trial cross-section of a slender reinforced rectangular column for the following data: \( b = 425 \text{ mm} \) (16.7 in.), \( h = 600 \text{ mm} \) (23.6 in.), \( f'_{c} = 32 \text{ MPa} \) (4640 psi), \( f_{sy} = 400 \text{ MPa} \) (58 ksi), \( A_s = A_{s'} = 2480 \text{ mm}^2 \) (3.84 sq.in.), \( d = 530 \text{ mm} \) (20.9 in.), \( d' = 70 \text{ mm} \) (2.7 in.), equivalent eccentricity, \( e = 90 \text{ mm} \) (3.5 in.), axial thrust due to dead load = 1250 kN (281 kip), axial thrust due to live load = 440 kN (99 kip), equivalent length, \( L_e = 10.8 \text{ m} \) (35.4 ft), \( E_c = 28500 \text{ MPa} \) (4133 ksi), \( E_s = 200 \times 10^3 \text{ MPa} \) (29000 ksi), \( e_{sy} = 0.002 \), and \( \phi_{cc} = 2.5 \).

For the example section, the usual strength calculations were performed. These calculations gave \( \phi P_{o} = 0.7 \times 8920 = 6244 \text{ kN} \) (1403 kip), \( \phi P_b = 0.7 \times 3014 = 2110 \text{ kN} \) (474 kip), and \( e_b = M_{b}/P_b = 321 \text{ mm} \) (12.6 in.). Also, \( P_u = (1.4 \times 1250) + (1.7 \times 440) = 2498 \text{ kN} \) (561 kip).

**Proposed Method**

(i) Calculate \( \Delta_{cp} \):

From Eq.(8), \( \lambda = 0.6 + (321/8 \times 90) = 1.04 > 1.0 \); therefore, take \( \lambda = 1.0 \). Also, \( (1 + 0.8 \times \phi_{cc}) = (1 + 0.8 \times 2.5) = 3.0 \) and \( I_g = 425 \times 600^3/12 \).
Substituting these values in Eqs. (6) and (10), we get \( P_c = 6149 \text{ kN} \) (1382 kip) and \( P_{co} = 18447 \text{ kN} \) (4146 kip).

Assume that the dead load is the only sustained load and \( P_\phi = 1250 \text{ kN} \) (281 kip).

From Eqs. (5) and (9),

\[
\Delta_{tot} = \frac{90}{[(6149/1250)-1]} = 23 \text{ mm (0.9 in.)}
\]

\[
\Delta_c = \frac{90}{[(18447/1250)-1]} = 6 \text{ mm (0.2 in.)}
\]

Hence, from Eq.(11), \( \Delta_{cp} = 23 - 6 = 17 \text{ mm (0.7 in.)} \).

(ii) Calculate \( \Delta_y \):

Because \( P_u > \phi P_b \), from Eqs.(15) and (17),

\[
\Delta_{yb} = (0.003+0.002)(10800)^2/(\pi^2\times530) = 112 \text{ mm (4.4 in.)}
\]

\[
\Delta_y = 112(6244-2498)/(6244-2110) = 101 \text{ mm (4.0 in.)}
\]

(iii) Calculate Magnified Factored Moment, \( M_c \):

From Eq.(14), \( \delta_c = 1 + \left(\frac{17+101}{90}\right) = 2.31 \)

From Eq.(13), \( M_c = 2498 \times 0.09 \times 2.31 = 519 \text{ kNm (59 in.–kip)} \).

From the strength calculations, we find that when \( P_u = \phi P_n = 2498 \text{ kN} \) (561 kip) the design bending strength, \( \phi M_n \), of the trial cross-section is 662 kNm (75 in.–kip). This is about 28 percent larger than the magnified factored moment. Therefore, the cross-section is more than adequate to carry the design loads.
ACI Method

\[ \beta_d = (1.4 \times 1250 \times 0.09)/(0.09(1.4 \times 1250 + 1.7 \times 440)) = 0.7 \]

\[ E_s I_{sc} = 200 \times 10^3 \times 2 \times 2480(300-70)^2 = 0.525 \times 10^{14} \]

\[ E_c I_g = 28500 \times 425 \times 600^3/12 = 2.18 \times 10^{14} \]

From the ACI Code Eq. (10–10),

\[ EI = \frac{(2.18 \times 10^{14}/5) + (0.525 \times 10^{14})}{1 + 0.7} = 0.565 \times 10^{14} \]

From Code Eq. (10–9),

\[ P_c = \frac{\pi^2}{(10800)^2} \times \frac{0.565 \times 10^{14}}{(10^3)} = 4780 \text{ kN (1074 kip)} \]

Substitution of this value in the Code Eq.(10–7) gives

\[ \delta_b = \frac{1}{1 - 2498/(0.7 \times 4780)} = 3.94 \]

Magnified factored moment = 3.94 \times 0.09 \times 2498 = 886 kNm (100 in–kip). This moment is about 34 percent larger than the available design bending strength \( \phi M_n = 662 \text{ kNm (75 in–kip)} \).

If EI is based on the Code Eq.(10–11), we get \( P_c = 4340 \text{ kN (975 kip)} \), \( \delta_b = 5.63 \), and the magnified factored moment = 1266 kNm (143 in–kip). This moment is about 91 percent larger than the available bending strength.

The above results, once again, confirm the conservative nature of the ACI Code method.
CONCLUDING REMARKS

A method for calculating the strength of reinforced concrete slender columns has been presented. It is based on the stability analysis of a standard pin-ended column as shown in Fig.1. The moment-curvature diagrams required in the analysis have been idealized as either elastic-plastic or elastic-brittle behaviour (Fig.2). The creep deflection, which is included in the analysis as an additional eccentricity, is calculated by Eqs. (5), (9), and (11).

The calculated strengths show good agreement with 80 test values available in the literature. The test data covered a wide range of parameters. The mean value of test/calculated is 1.06 with a coefficient of variation of 13 percent.

The method has been compared with the design provisions contained in Section 10.11 of the ACI Code (Ref.1). The Code method is significantly conservative for columns with larger slenderness ratio and smaller eccentricity.

Based on the method, proposals for design have been made. These proposals are considered to be more rational than the Code approach and yet simple to use in routine design calculations. An example has been presented to illustrate the application of the design proposals.

NOTATION

\[ A_g = \] gross cross-sectional area of a column

\[ A_s = \] area of tensile steel

\[ A_s' = \] area of compressive steel

\[ b = \] width of a cross-section

\[ C_m = \] a factor given by Eq.(10–12) of the ACI Code
\begin{align*}
  d &= \text{effective depth of a cross-section} \\
  d' &= \text{depth of compressive steel from compression face} \\
  E_c &= \text{modulus of elasticity of concrete} \\
  E_s &= \text{modulus of elasticity of steel} \\
  EI &= \text{effective bending stiffness} \\
  e &= \text{eccentricity of axial thrust, measured from the plastic centroid} \\
  e_b &= \text{value of } e \text{ at balanced failure conditions } (= M_b/P_b) \\
  f'_c &= \text{compressive strength of concrete} \\
  f_{sy} &= \text{yield strength of reinforcing steel} \\
  h &= \text{overall depth of a cross-section in the plane of bending} \\
  I_g &= \text{moment of inertia for the gross concrete section} \\
  L_e &= \text{effective length of a column} \\
  M_b &= \text{particular ultimate strength in combined axial compression and bending at balanced failure conditions} \\
  M_{2b} &= \text{value of larger factored end moment} \\
  M_c &= \text{magnified factored moment} \\
  M_{\text{max}} &= \text{maximum moment at which stability failure occurs (Fig.2)}
\end{align*}
\[ M_n \] = ultimate strength in combined axial compression and bending

\[ P_b \] = particular axial load strength at balanced failure conditions

\[ P_{c \cdot P_{co}} \] = critical loads

\[ P_{\text{max}} \] = axial thrust at which stability failure occurs (Fig.2)

\[ P_n \] = axial load strength at given eccentricity

\[ P_o \] = axial load strength at zero eccentricity

\[ P_\phi \] = axial thrust due to sustained loads

\[ r \] = radius of gyration of cross-section of a column

\[ \beta_d \] = factor as defined in the ACI Code

\[ \Delta \] = lateral deflection of a column

\[ \Delta_{cp} \] = creep deflection

\[ \Delta_e \] = elastic component of the total deflection

\[ \Delta_{\text{tot}} \] = total deflection

\[ \Delta_y \] = deflection at \( M_{\text{max}} \) (Fig.2)

\[ \Delta_{yb} \] = particular value of \( \Delta_y \) at balanced failure conditions in combined axial compression and bending
\[ \Delta_{yo} \] = particular value of \( \Delta_y \) in pure bending when the tensile steel just yields

\[ \delta_b \] = moment magnifier

\[ \lambda \] = a factor (Eq.8)

\[ \varepsilon_{sy} \] = yield strain of steel

\[ \kappa \] = curvature of column segment

\[ \kappa_y \] = curvature at \( M_{max} \)

\[ \phi \] = strength reduction factor

\[ \phi_{cc} \] = creep factor

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TABLE 1 — COMPARISON OF TEST AND CALCULATED STRENGTHS OF SLENDER COLUMNS TESTED BY GOYAL AND JACKSON (Ref.6)

<table>
<thead>
<tr>
<th>Column Reference</th>
<th>$P_{max}$ (kips)</th>
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Table 1 (Continued)

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<th>Test Calculated</th>
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Test:
- Number of columns = 46
- Mean value = 1.02
- Coefficient of variation = 10 percent

Note: 1 kip = 4.45 kN.
### TABLE 2 – COMPARISON OF TEST AND CALCULATED STRENGTHS OF SLENDER COLUMNS TESTED BY KORDINA (REF. 7)

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<tr>
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<td>220</td>
<td>1.16</td>
</tr>
<tr>
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<td>201</td>
<td>164</td>
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<tr>
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<td>519</td>
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<tr>
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<tr>
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<td>213</td>
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</table>

* ignored in the mean value

Number of columns = 12

Test: Mean value = 1.08
Calculated: Coefficient of variation = 15 percent

Note: 1 kip = 4.45 kN.
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<th>Calculated</th>
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<th>Calculated</th>
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*ignored in the mean value

Test: Number of columns = 22
Mean value = 1.14

Calculated: Coefficient of variation = 13 percent

Note: 1 kip = 4.45 kN.
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<th>Coefficient of variation</th>
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</tr>
<tr>
<td>Kordina (Ref.7)</td>
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<td>Ramu et al (Ref.5)</td>
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<tr>
<td>All results</td>
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</table>
Fig. 1 Standard Pin-Ended Column.
e = eccentricity
$\Delta_{cp}$ = creep deflection due to sustained load $P_{\phi}$
$\Delta_y$ = deflection at $M_{max}$
$P_{max}$ = axial thrust at failure
$M_{max} = P_{max} (e + \Delta_{cp} + \Delta_y)$
Fig. 2 Simplified Stability Analysis.
Fig. 3 Comparison of Load Carrying Capacity of Slender Columns.